

HOMOGENEOUS G-ALGEBRAS II

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ABSTRACT. G-algebras, or Groebner bases algebras, were considered by Levandovsky in, [Le], these algebras include very important families of algebras, like the Weyl algebras [Co], [Miℓ] and the universal enveloping algebra of a finite dimensional Lie algebra. These algebras are not graded in a natural way, but they are filtered. Going to the associated graded algebra is the standard way to study their properties [Miℓ]. In a previous paper [MMo1] we studied an homogeneous version of the G-algebras, which results naturally graded, and through a des homogenization property, we obtain an ordinary G-algebra. The main results in our first paper were the following:

Homogeneous G-algebras are Koszul of finite global dimension, Artin Schelter regular, noetherian and have a Poincare Birkoff basis. We give the structure of their Yoneda algebras by generators and relations.

In the second part of the paper we consider the quantum polynomial ring and prove, for the finitely generated graded modules, cohomology formulas analogous to those we have in the usual polynomial ring.

In the third part we use the results of the previous part to study the cohomology of the homogeneous G-algebras.

Part four was dedicated to the study of the relations between an homogeneous G-algebra B and its des homogenization $B/(Z-1)B = A$, Applications to the study of finite dimensional Lie algebras and Weyl algebras were given.

In the last sections of the paper we study the relations, at the level of modules, among the homogenized algebra B , its graded localization B_z , the des homogenization $A=B/(Z-1)B$ and the Yoneda algebra $B^!$.

In this paper we investigate the relations among all these algebras at the level of derived categories.

1. SOME RESULTS ON THE HOMOGENEOUS G- ALGEBRAS

Let \mathbb{k} be a field of zero characteristic. A G- algebra A_n has the following description by generators and relations: $A_n = \mathbb{k} \langle X_1, X_2, \dots, X_n \rangle / \langle X_i X_j - b_{ij} X_j X_i + \sum_{k=1}^n c_{ij}^k X_k \rangle + d_{ij} \mid i > j, b_{ij} \in \mathbb{k} - \{0\} \rangle$, where $\mathbb{k} \langle X_1, X_2, \dots, X_n \rangle$ is the free algebra in n generators.

Examples of these algebras are the following:

- i) The Weyl algebras $D_n = \mathbb{k} \langle X_1, X_2, \dots, X_n, \partial_1, \partial_2, \dots, \partial_n \rangle / I$, with $I = \langle [X_i, \partial_j] - \delta_{ij}, [X_i, X_j], [\partial_i, \partial_j] \rangle$, and δ_{ij} Kronecker's delta.
- ii) Let \mathfrak{g} a finite dimensional Lie algebra with basis $\mu_1, \mu_2, \dots, \mu_n$ and $[\mu_i, \mu_j] = \sum c_{i,j}^k \mu_k$. The Universal enveloping algebra is the algebra $U(\mathfrak{g}) = \mathbb{k} \langle X_1, X_2, \dots, X_n, \rangle / I$, with the ideal $I = \langle X_i X_j - X_j X_i - \sum_{k=1}^n c_{ij}^k X_k \rangle$.
- iii) Quantized versions of i), ii).

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We know that G-algebras are noetherian, they have a Poincare-Birkoff basis, which induces a natural filtration, the associated graded algebra is the quantum polynomial ring.

In the particular example of the Weyl algebras, we know that in addition they are simple algebras.

It is of interest to study both, the category of finitely generated A_n -modules, and the derived category $D^b(\text{mod}_{A_n})$. This is not a simple task since for examples i), ii) we do not in general know even the irreducible A_n -modules, so what we do instead is to associate to A_n the graded algebra B_n defined by generators and relations as $B_n = \mathbb{k} \langle X_1, X_2, \dots, X_n, Z \rangle / \langle X_i X_j - b_{ij} X_j X_i + \sum_{k=1}^n c_{ij}^k Z X_k + d_{ij} Z^2 \mid i > j, b_{ij} \in \mathbb{k} - \{0\}, X_i Z - Z X_i \rangle$, where $\mathbb{k} \langle X_1, X_2, \dots, X_n, Z \rangle$ is the free algebra in n generators. The algebra B_n , thus defined will be called the homogenization of A_n and we recover A_n by the deshomogenization $A_n \cong B_n / (Z-1)B_n$. We call algebras like B_n , homogeneous G-algebras.

Homogeneous G-algebras are much easier to study than G-algebras, since, as we proved in our first paper, they are Koszul of finite global dimension and Artin Schelter regular [AS], by Koszul theory [GM1], [GM2], [M], [Sm] its Yoneda algebra $B_n^!$ is a finite dimensional selfinjective algebra, and there is an equivalence of triangulated categories $D^b(\text{gr}_{B_n^!}) \cong D^b(\text{gr}_{B_n})$ and a duality between the categories of left B_n and $B_n^!$ Koszul modules, respectively, $F : K_{B_n} \rightarrow K_{B_n^!}$. In order to understand the A_n -modules via the finite dimensional algebra $B_n^!$, we must study the relations between the categories mod_{A_n} and $\text{gr}_{B_n^!}$. In [MMo1] we concentrated in the structure of the algebras B_n and $B_n^!$ and studied the relations among the categories mod_{A_n} , gr_{B_n} and $\text{gr}_{B_n^!}$. Here we study the relations among the algebras A_n , B_n and $B_n^!$ at the level of derived categories.

For the benefit of the reader we start recalling some of the results in our first paper [MMo1]. The homogeneous G-algebra B_n has a Poincare Birkoff basis, B_n is two sided noetherian of both, global and Gelfand Kirillov dimension $n+1$. The polynomial algebra $\mathbb{k}[Z]$ is contained in the center of B_n , and B_n is a finitely generated $\mathbb{k}[Z]$ -algebra. Denote by $C_n = \mathbb{k}_q[X_1, X_2, \dots, X_n]$ the quantum polynomial algebra C_n is a Koszul algebra with Yoneda algebra the quantized exterior algebra $C_n^! = \mathbb{k}_q[X_1, X_2, \dots, X_n] / \langle X_i^2 \rangle$, these algebras are related to B_n and $B_n^!$ as follows: C_n is isomorphic to the quotient B_n / ZB_n and $C_n^!$ is a subalgebra of $B_n^!$. Moreover, there is a left (right) $C_n^!$ -module decomposition $B_n^! = C_n^! \oplus C_n^! Z$ ($B_n^! = C_n^! \oplus ZC_n^!$).

As we mentioned above, the relation of B_n with the G-algebra A_n is given by the isomorphism $B_n / (Z-1)B_n \cong A_n$.

Since Z is an element in the center of B_n , we take the graded localization $(B_n)_Z$. The algebra $(B_n)_Z$ is a strongly graded \mathbb{Z} -graded algebra which has in degree zero an algebra $((B_n)_Z)_0$ isomorphic to $B_n / (Z-1)B_n$, which is in turn isomorphic to $B_n / (Z-1)B_n \cong A_n$. By Dade's theorem, [Da] there is an exact equivalence of categories $\text{Gr}_{(B_n)_Z} \cong \text{Mod}_{A_n}$, which induces by restriction an exact equivalence of categories between the corresponding categories of finitely generated modules $\text{gr}_{(B_n)_Z} \cong \text{mod}_{A_n}$. In view of the previous statements, to understand the module structure over a G-algebra A_n it is enough to study the relations between gr_{B_n} and the finitely generated $(B_n)_Z$ -modules, $\text{gr}_{(B_n)_Z}$, but this is just the localization

process, which is rather known, then to use Koszul duality to relate gr_{B_n} and $\text{gr}_{B_n^!}$.

We recall some basic facts about selfinjective finite dimensional algebras that will be needed, in the particular situation we are considering. For more details we refer to the paper by Yamagata [Y] .

Let Λ be a finite dimensional selfinjective algebra over a field \mathbb{k} . Denote by $D(\Lambda) = \text{Hom}_{\mathbb{k}}(\Lambda, \mathbb{k})$ the standard bimodule. There is an isomorphism of left Λ -modules $\varphi: \Lambda \rightarrow D(\Lambda)$, which induces by adjunction a map $\beta': \Lambda \otimes_{\Lambda} \Lambda \rightarrow \mathbb{k}$. By definition, $\beta'(a \otimes b) = \varphi(b)(a)$. The composition: $\Lambda \times \Lambda \xrightarrow{p} \Lambda \otimes_{\Lambda} \Lambda \xrightarrow{\beta'} \mathbb{k}$, $\beta = \beta'p$ is a non degenerated Λ -bilinear form, and $\Lambda \otimes_{\Lambda} \Lambda$ is the cokernel of the map $\Lambda \otimes_{\mathbb{k}} \Lambda \otimes_{\mathbb{k}} \Lambda \rightarrow \Lambda \otimes_{\mathbb{k}} \Lambda$ given by $a \otimes b \otimes c \rightarrow ab \otimes c - a \otimes bc$. Let $\pi: \Lambda \otimes_{\mathbb{k}} \Lambda \rightarrow \Lambda \otimes_{\Lambda} \Lambda$ be the cokernel map.

The map β' induces a map $\bar{\beta}: \Lambda \otimes_{\mathbb{k}} \Lambda \rightarrow \mathbb{k}$ by $\bar{\beta}(x \otimes y) = \beta'(y \otimes x)$. Hence $\bar{\beta}$ is also non degenerated. In consequence, there is a \mathbb{k} -linear isomorphism $\psi: \Lambda \rightarrow D(\Lambda)$ given by: $\psi(a_1)(a_2) = \bar{\beta}(a_2 \otimes a_1) = \beta(a_1 \otimes a_2) = \varphi(a_2)(a_1)$.

Set $\sigma = \psi^{-1}\varphi$. There is a chain of equalities:

$$\beta(\sigma(y), x) = \beta(\psi^{-1}\varphi(y), x) = \psi\psi^{-1}\varphi(y)(x) = \varphi(y)(x) = \bar{\beta}(y \otimes x).$$

The map $\sigma: \Lambda \rightarrow \Lambda$ is an isomorphism of \mathbb{k} -algebras.

$$\begin{aligned} \beta(\sigma(y_1 y_2) \otimes z) &= \beta(z \otimes y_1 y_2) = \beta(z y_1 \otimes y_2) = \bar{\beta}(y_2, z y_1) = \beta(\sigma(y_2), z y_1) = \beta(\sigma(y_2) z, y_1) \\ &= \bar{\beta}(y_1, \sigma(y_2) z) = \beta(\sigma(y_1), \sigma(y_2) z) = \beta(\sigma(y_1) \sigma(y_2), z). \end{aligned}$$

Since z is arbitrary and β non degenerated $\sigma(y_1 y_2) = \sigma(y_1) \sigma(y_2)$.

Let $D(\Lambda)_{\sigma} (\sigma^{-1} D(\Lambda))$ be the Λ - Λ bimodule with right (left) multiplication shifted by σ (σ^{-1}). Then $\varphi: \Lambda \rightarrow D(\Lambda)_{\sigma}$ and $\psi: \Lambda \rightarrow \sigma^{-1} D(\Lambda)$ are isomorphisms of Λ - Λ bimodules.

$$\varphi(xa)(y) = \beta(y, xa) = \beta(\sigma(x)\sigma(a), y) = \beta(\sigma(x), \sigma(a)y) = \beta(\sigma(a)y, x) =$$

$$\varphi(x)(\sigma(a)y) = \varphi(x)\sigma(a)(y), \text{ for all } y. \text{ Therefore } \varphi(xa) = \varphi(x)\sigma(a). \text{ As claimed.}$$

$$\text{In a similar way, } \psi(xb)(y) = \varphi(y)(xb) = \beta(xb, y) = \beta(x, by) = \varphi(by)(x) = \psi(x)b(y).$$

Since y is arbitrary, $\psi(xb) = \psi(x)b$.

$$\text{In the other hand, } \varphi(y)(x) = (\psi(x)(y)) = \beta(x, y) = \beta(\sigma\sigma^{-1}(x), y) = \beta(y, \sigma^{-1}(x)) = \varphi(\sigma^{-1}(x))(y). \text{ Hence, } \psi(x) = \varphi(\sigma^{-1}(x)).$$

$$\text{It follows: } \psi(bx) = \varphi(\sigma^{-1}(b)\sigma^{-1}(x)) = \sigma^{-1}(b)\varphi(\sigma^{-1}(x)) = \sigma^{-1}(b)\psi(x).$$

Let M be a finitely generated Λ -module. Since ${}_{\sigma}\Lambda \cong D(\Lambda)$ as bimodule, there are natural isomorphisms:

$$D(M^*) = \text{Hom}_{\Lambda}(\text{Hom}_{\Lambda}(M, \Lambda), D(\Lambda)) = \text{Hom}_{\Lambda}(\text{Hom}_{\Lambda}(M, \Lambda), {}_{\sigma}\Lambda) \cong \sigma M^{**} \cong \sigma M, \text{ where } \sigma M = M \text{ as abelian group and multiplication by } \Lambda \text{ shifted by } \sigma.$$

We look now to the case Λ a positively graded selfinjective \mathbb{k} -algebra and $\varphi: \Lambda \rightarrow D(\Lambda)[n]$ an isomorphism of graded Λ -modules. Let a, x be elements of Λ of degrees k and j , respectively. Then $\varphi(xa) = \varphi(x)\sigma(a)$ is an homogeneous element of degree $k+j$ and $\sigma(a) = \sum \sigma(a)_i$, with $\sigma(a)_i$ homogeneous elements of degree i . Hence $\varphi(x)\sigma(a)_i = 0$ for all $i \neq j$. But $\varphi(x)\sigma(a)_i(1) = \varphi(x)(\sigma(a)_i) = \beta(\sigma(a)_i, x) = 0$ for all homogeneous elements x . Hence $\beta(\sigma(a)_i, \Lambda) = 0$ and $\sigma(a)_i = 0$ for $i \neq j$.

We have proved σ is an isomorphism of graded \mathbb{k} -algebras, which induces isomorphisms of graded Λ - Λ bimodules: $\varphi: \Lambda \rightarrow D(\Lambda)_{\sigma}[n]$, and $\psi: \Lambda \rightarrow \sigma^{-1} D(\Lambda)[n]$.

We only need to check ψ is an isomorphism of graded \mathbb{k} -vector spaces.

Being φ a graded map, $\varphi = \{\varphi_{\ell}\}$ and $\varphi_{\ell}: \Lambda_{\ell} \rightarrow \text{Hom}_{\mathbb{k}}(\Lambda_{n-\ell}, \mathbb{k})[n]$ isomorphisms, inducing maps $\beta'_{\ell}: \Lambda_{n-\ell} \otimes \Lambda_{\ell} \rightarrow \mathbb{k}$ and $\bar{\beta}_{\ell}: \Lambda_{\ell} \otimes_{\mathbb{k}} \Lambda_{n-\ell} \rightarrow \mathbb{k}$, with $\bar{\beta}_{\ell}(x \otimes y) = \beta'_{\ell}(y \otimes x)$.

Each $\overline{\beta}_\ell$ induces maps $\psi_\ell \Lambda_{:n-\ell} \rightarrow D(\Lambda_\ell)[n]$ such that $\psi = \{\psi_\ell\}$ becomes a graded map.

In case M is a finitely generated graded left Λ -module there is a chain of isomorphisms of graded Λ -modules:

$$D(M^*) = \text{Hom}_\Lambda(\text{Hom}_A(M, \Lambda), D(\Lambda)) = \text{Hom}_\Lambda(\text{Hom}_\Lambda(M, \Lambda), {}_\sigma\Lambda[-n]) \cong {}_\sigma M^{**}[-n] \cong {}_\sigma M[-n].$$

Assume now Λ is Koszul sefinjective with Nakayama automorphism σ and Yoneda algebra Γ . It was remarked in [M] that under these conditions there is natural action of σ as a graded automorphism of Γ , we will recall now this construction.

Let x be an element of $\text{Ext}_\Lambda^n(\Lambda_0, \Lambda_0) = \bigoplus_{i,j} \text{Ext}_\Lambda^n(S_i, S_j)$, $x = (x_{i,j})$ with $x_{i,j}$ the extension:

$$x_{i,j}: 0 \rightarrow S_j[-n] \rightarrow E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_n \rightarrow S_i \rightarrow 0.$$

Then $\sigma x_{i,j}$ is the extension:

$$\sigma x_{i,j}: 0 \rightarrow \sigma S_j[-n] \rightarrow \sigma E_1 \rightarrow \sigma E_2 \rightarrow \dots \rightarrow \sigma E_n \rightarrow \sigma S_i \rightarrow 0.$$

Since σ is a permutation of the graded simple, $\sigma x = (\sigma x_{i,j})$ is an element of $\text{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)$ and $\sigma: \text{Ext}_\Lambda^n(\Lambda_0, \Lambda_0) \rightarrow \text{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)$ is an isomorphism of \mathbb{k} -vector spaces which extends to a graded automorphism of $\Gamma = \bigoplus_{n \geq 0} \text{Ext}_\Lambda^n(\Lambda_0, \Lambda_0)$.

Let M be a finitely generated (graded) A -module and $x = (x_j) \in \text{Ext}^n(M, \Lambda_0) = \bigoplus_{j \geq 0} \text{Ext}_\Lambda^n(M, S_j)$.

$$x_j: 0 \rightarrow S_j[-n] \rightarrow E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_n \rightarrow M \rightarrow 0.$$

Then $\sigma x = (\sigma x_j)$ with

$$\sigma x_j: 0 \rightarrow \sigma S_j[-n] \rightarrow \sigma E_1 \rightarrow \sigma E_2 \rightarrow \dots \rightarrow \sigma E_n \rightarrow \sigma M \rightarrow 0.$$

In this case there is an isomorphism of \mathbb{k} -vector spaces:

$$\sigma: \text{Ext}_\Lambda^n(M, \Lambda_0) \rightarrow \text{Ext}_\Lambda^n(\sigma M, \Lambda_0), \text{ which induces a graded isomorphism: } \sigma: \bigoplus_{n \geq 0} \text{Ext}_\Lambda^n(M, \Lambda_0) \rightarrow \bigoplus_{n \geq 0} \text{Ext}_\Lambda^n(\sigma M, \Lambda_0).$$

We also call the Nakayama automorphism to the automorphism σ of Γ .

We now look in more detail to the Nakayama automorphism σ of the shriek algebra $B_n^!$ of the homogeneous algebra B_n .

The graded ring $B_n^!$ has a decomposition: $(B_n^!)_0 = (C_n^!)_0$, $(B_n^!)_1 = (C_n^!)_1 \oplus \mathbb{Z}(C_n^!)_0$, $(B_n^!)_2 = (C_n^!)_2 \oplus \mathbb{Z}(C_n^!)_1$, ..., $(B_n^!)_i = (C_n^!)_i \oplus \mathbb{Z}(C_n^!)_{i-1}, \dots$, $(B_n^!)_n = (C_n^!)_n \oplus \mathbb{Z}(C_n^!)_{n-1}$, $(B_n^!)_{n+1} = \mathbb{Z}(C_n^!)_n$.

The algebra $C_n^!$ is the quantized exterior algebra in n variables, hence

$$\dim_{\mathbb{k}}(C_n^!)_j = \binom{n}{j} = \binom{n}{n-j} = \dim_{\mathbb{k}}(C_n^!)_{n-j}.$$

Since $(B_n^!)_j = (C_n^!)_j \oplus \mathbb{Z}(C_n^!)_{j-1}$, it follows:

$$\dim_{\mathbb{k}}(B_n^!)_j = \binom{n}{j} + \binom{n}{j-1} = \binom{n}{n+1-j} + \binom{n}{n-j} = \dim_{\mathbb{k}}(B_n^!)_{n+1-j}.$$

The graded left module $D(B_n^!)$ decomposes in homogeneous components:

$$D(B_n^!) = D((B_n^!)_{n+1}) + D((B_n^!)_n) + \dots + D((B_n^!)_0).$$

Each component $(B_n^!)_i$ has as basis paths of length i , either of the form

$X_{j_1} X_{j_2} \dots X_{j_{i-1}} Z$, or of the form $X_{j_1} X_{j_2} \dots X_{j_i}$, and $D((B_n^!)_{n+1-i})$ has as basis the dual basis of paths of length $n+1-i$.

The isomorphism of graded left modules: $\varphi: B_n^! \rightarrow D(B_n^!)[-n-1]$, sends a path of, either of the form $\gamma = X_{j_1} X_{j_2} \dots X_{j_{i-1}} Z$, or of the form

$\gamma = X_{j_1} X_{j_2} \dots X_{j_i}$, to the dual basis $f_{\rho-\gamma}$ of the path $\rho-\gamma$ of length $n+1-i$, with ρ the path of maximal length $\rho = X_1 X_2 \dots X_n Z$.

Since $(B_n^!)_i = (C_n^!)_i \oplus Z(C_n^!)_{i-1}$, the isomorphism ϕ restricts to isomorphisms of \mathbb{k} -vector spaces $\varphi: (C_n^!)_i \rightarrow D(Z(C_n^!)_{n-i-1})$ and $\varphi: (ZC_n^!)_{i-1} \rightarrow D((C_n^!)_{n-i})$, hence, φ induces isomorphisms of $C_n^!$ -modules $\varphi: (C_n^!) \rightarrow D(Z(C_n^!))$ and $\varphi: (ZC_n^!) \rightarrow D((C_n^!))$.

Now the isomorphism $\psi: B_n^! \rightarrow D(B_n^!)[-n-1]$ given $\psi(b_1)(b_2) = \varphi(b_2)(b_1)$ is such that for $c_1 \in (C_n^!)_i$ and $b \in B_n$, $b = \sum_{i=0}^{n+1} b_i$, $\psi(c_1)(b) = \sum_{k=0}^{n+1} \varphi(b_k)(c_1)$, since for all $k \neq i$ the length of b_k is different from $n+1-i$, then $\varphi(b_k)(c_1) = 0$, $\psi(c_1) \in D(Z(C_n^!)_{n-i-1})$, hence, ψ induces an isomorphism of graded $C_n^!$ -modules $\psi: (C_n^!) \rightarrow D(Z(C_n^!))$ and in a similar way an isomorphism $\psi: (ZC_n^!) \rightarrow D((C_n^!))$. It follows the Nakayama automorphism σ restricts to an automorphisms of graded rings: $\sigma: C_n^! \rightarrow C_n^!$ and of $C_n^!$ -bimodules $\sigma: ZC_n^! \rightarrow ZC_n^!$.

Any automorphism σ of a ring Λ takes the center to the center, since $z \in Z(\Lambda)$ implies that for any $b \in \Lambda$, $\sigma(Zb) = \sigma(Z)\sigma(b) = \sigma(bZ) = \sigma(b)\sigma(Z)$.

In case Λ is the homogeneous Weyl algebra it is not difficult to see that the center of Λ is $\mathbb{k}[Z]$ and since $\sigma(Z)$ is an homogeneous element of degree one in $Z(B) = \mathbb{k}[Z]$, it is clear $\sigma(Z) = kZ$ with k a non zero element of the field \mathbb{k} .

2. THE DERIVED CATEGORIES $D^b(\text{Qgr}_{B_n})$ AND $D^b(\text{gr}_{(B_n)_Z})$.

We describe now the main results of the paper: let's consider a homogeneous G-algebra B_n , and let Qgr_{B_n} be the quotient category of gr_{B_n} , module the objects of finite length, this is: the category of "tails" consider in [AZ],[G], [P]. Qgr_{B_n} is an abelian category and there is an exact functor $\pi: \text{gr}_{B_n} \rightarrow \text{Qgr}_{B_n}$. It was proved in [MM],[MS] that there is an exact duality $\underline{\text{gr}}_{B_n^!} \cong D^b(\text{Qgr}_{B_n})$, with $D^b(\text{Qgr}_{B_n})$ the derived category of bounded complex of objects in Qgr_{B_n} , we must consider the relations between the categories $D^b(\text{Qgr}_{B_n})$ and $D^b(\text{gr}_{(B_n)_Z})$, which in view of the results of [MMo], the last one is equivalent as triangulated categories to $D^b(\text{mod}_{A_n})$. We prove that there exists an exact dense functor: $D^b(\psi): D^b(\text{Qgr}_{B_n}) \rightarrow D^b(\text{gr}_{(B_n)_Z})$ with kernel $\mathcal{T} = \text{Ker} D^b(\psi) = \{\pi(M^\circ) \mid M^\circ \in D^b(\text{gr}_{B_n}) \text{ for all } i, H^i(M^\circ) \text{ is of } Z\text{-torsion}\}$. The main theorem of the section is that there exists an equivalence of triangulated categories $D^b(\text{Qgr}_{B_n}) / \mathcal{T} \cong D^b(\text{gr}_{(B_n)_Z})$, where the category \mathcal{T} is an "epasse" subcategory of $D^b(\text{Qgr}_{B_n})$ and $D^b(\text{Qgr}_{B_n}) / \mathcal{T}$ is the Verdier quotient. [Mi]

In section three, together with the quotient of categories considered in section two, we consider a full embedding of a subcategory \mathcal{F} of $D^b(\text{Qgr}_{B_n})$ in $D^b(\text{gr}_{(B_n)_Z})$. Here \mathcal{F} is the full subcategory of $D^b(\text{Qgr}_{B_n})$ consisting of the \mathcal{T} -local objects [Mi], this is: $\mathcal{F} = \{X^\circ \mid \text{Hom}_{D^b(\text{Qgr}_{B_n})}(\mathcal{T}, X^\circ) = 0\}$.

Using the duality $\bar{\phi}: \underline{\text{gr}}_{B_n^!} \rightarrow D^b(\text{Qgr}_{B_n})$ we obtain a pair of triangulated subcategories $(\mathcal{F}', \mathcal{T}')$ of $\underline{\text{gr}}_{B_n^!}$ such that $\mathcal{F}' \rightarrow \mathcal{F}$ and $\mathcal{T}' \rightarrow \mathcal{T}$ under the duality $\bar{\phi}$. We obtain the following characterization of the subcategories \mathcal{T}' and \mathcal{F}' of $\underline{\text{gr}}_{B_n^!}$: \mathcal{T}' is the smallest triangulated subcategories of $\underline{\text{gr}}_{B_n^!}$ containing the induced modules $M \otimes_{C_n^!} B_n^!$ and closed under the Nakayama automorphism, \mathcal{T}' has Auslander-Reiten triangles and they are of the type $\mathbb{Z}A_\infty$. For the category \mathcal{F}' we obtain the following characterization: \mathcal{F}' consists of the graded finitely generated $B_n^!$ -modules whose restriction to $C_n^!$ is injective. Furthermore, the category \mathcal{F}' is closed under

the Nakayama automorphism, it has Auslander-Reiten triangles and they are of type $\mathbb{Z}A_\infty$.

In order to obtain an equivalence instead of a duality we apply the usual duality $D: \underline{\text{gr}}_{B_n^!} \rightarrow \underline{\text{gr}}_{B_n^{\text{lop}}}$ to obtain subcategories T and F of $\underline{\text{gr}}_{B_n^{\text{lop}}}$ with $T=D(\mathcal{T}')$ and $F=D(\mathcal{F}')$ where T is the smallest triangulated subcategory of $\underline{\text{gr}}_{B_n^{\text{lop}}}$ containing the induced modules and F is the full subcategory of $\underline{\text{gr}}_{B_n^{\text{lop}}}$ consisting of those modules whose restriction to $C_n^!$ is projective.

Finally we obtain the main results of the paper: there is an equivalence of triangulated categories $\underline{\text{gr}}_{B_n^{\text{lop}}} / T \cong D^b(\text{mod}_{A_n})$ and there is a full embedding of triangulated categories $F \rightarrow D^b(\text{mod}_{A_n})$.

We start recalling the construction of the categories of "tails" QGr_{B_n} and Qgr_{B_n} .

Let M be a graded B_n -module, $t(M) = \sum_{L \in J} L$ and $J = \{L \mid \text{sub module of } M \text{ with } \dim_{\mathbb{k}} L < \infty\}$.

Claim: $t(M/t(M))=0$.

Let N be a finitely generated sub module of M such that $N+t(M)/t(M) = N/N \cap t(M)$ is finite dimensional over \mathbb{k} . Since B_n is noetherian $N \cap t(M)$ is a finitely generated submodule of $t(M)$, hence of finite dimension over \mathbb{k} . It follows N is finite dimensional, so $N \subset t(M)$.

Let N be an arbitrary sub module of M with $N+t(M)/t(M)$ finite dimensional over \mathbb{k} , then $N = \sum N_i$, with N_i finitely generated, each $N_i+t(M)/t(M)$ is finite dimensional, therefore $N_i \subset t(M)$. It follows $N \subset t(M)$ and t is an idempotent radical.

If we denote by gr_{B_n} and $\text{gr}_{(B_n)_Z}$ the categories of finitely generated graded B_n and $(B_n)_Z$ -modules, respectively, then the localization functor Q restricts to a functor $Q: \text{gr}_{B_n} \rightarrow \text{gr}_{(B_n)_Z}$.

Definition 1. Given a B_n -module M , define the Z -torsion of M as: $t_Z(M) = \{m \in M \mid \text{there exists } n > 0 \text{ with } Z^n m = 0\}$. It is clear t_Z is an idempotent radical. We say M is Z -torsion when $t_Z(M) = M$ and Z -torsion free if $t_Z(M) = 0$.

The kernel of the natural map $M \rightarrow M_Z$ is $t_Z(M)$.

Definition 2. We say that a (graded) B_n -module is torsion if $t(M) = M$ and torsion free if $t(M) = 0$.

It is clear $t(M)$ is Z -torsion and $t(M) \subset t_Z(M)$. Therefore if M is torsion then it is Z -torsion and if M is Z -torsion free then it is torsion free.

The torsion free modules form a Serre (or thick) subcategory of Gr_{B_n} , we localize with respect to this subcategory as explained in [Ga], [P]. Denote by QGr_{B_n} the quotient category and let $\pi: \text{Gr}_{B_n} \rightarrow \text{QGr}_{B_n}$ be the quotient functor, $\text{QGr}_{B_n} = \text{Gr}_{B_n} / \text{Torsion}$, is an abelian category with enough injectives and π is an exact functor. When taking this quotient we are inverting the maps of B_n -graded modules, f: $M \rightarrow N$ such that $\text{Ker} f$ and $\text{Coker} f$ are torsion.

The category QGr_{B_n} has the same objects as Gr_{B_n} and maps:

$\text{Hom}_{\text{QGr}_{B_n}}(\pi(M), \pi(N)) = \varinjlim \text{Hom}_{\text{Gr}_{B_n}}(M', N/t(N))$, the limit running through all the sub modules M' of M such that M/M' is torsion.

If M is a finitely generated module, then the limit has a simpler form:

$$\text{Hom}_{\text{QGr}_{B_n}}(\pi(M), \pi(N)) = \varinjlim_{\mathbb{k}} \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}, N/t(N)).$$

In case N is torsion free: $\text{Hom}_{\text{QGr}_{B_n}}(\pi(M), \pi(N)) = \varinjlim_k \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}, N)$.

The functor $\pi: \text{Gr}_{B_n} \rightarrow \text{QGr}_{B_n}$ has a right adjoint: $\varpi: \text{QGr}_{B_n} \rightarrow \text{Gr}_{B_n}$ such that $\pi\varpi \cong 1$. [P].

If we denote by gr_{B_n} the category of finitely generated graded B_n -modules, and by Qgr_{B_n} the full subcategory of QGr_{B_n} consisting of the objects $\pi(N)$ with N finitely generated, then the functor π induces by restriction a functor: $\pi: \text{gr}_{B_n} \rightarrow \text{Qgr}_{B_n}$. The kernel of π is: $\text{Ker}\pi = \{M \in \text{gr}_{B_n} \mid \pi(M) = 0\} = \{M \in \text{gr}_{B_n} \mid t(M) = M\}$.

In the other hand, the functor

$Q = (B_n)_Z \otimes_B -: \text{gr}_{B_n} \rightarrow \text{gr}_{(B_n)_Z}$ has kernel $\{M \in \text{gr}_{B_n} \mid M_Z = 0\} = \{M \in \text{gr}_{B_n} \mid t_Z(M) = M\}$.

It follows: $\text{Ker}\pi \subset \text{Ker}((B_n)_Z \otimes_B -)$.

According to [P] (pag. 173 Cor. 3.11) there exists a unique functor ψ such that the following diagram commutes:

$$\begin{array}{ccc} \text{gr}_{B_n} & \xrightarrow{\pi} & \text{Qgr}_{B_n} \\ (B_n)_Z \otimes_B - & \searrow & \swarrow \psi \\ & \text{gr}_{(B_n)_Z} & \end{array}$$

This is: $\psi\pi = (B_n)_Z \otimes_B -$.

Proposition 1. *The functor $\psi: \text{Qgr}_{B_n} \rightarrow \text{gr}_{(B_n)_Z}$ is exact.*

Proof. Let $0 \rightarrow \pi(M) \xrightarrow{\hat{f}} \pi(N) \xrightarrow{\hat{g}} \pi(L) \rightarrow 0$ be an exact sequence in Qgr_{B_n} . We may assume M, N, L torsion free. Then::

$$\hat{f} \in \varinjlim_k \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}, N) \text{ and } \hat{g} \in \varinjlim_s \text{Hom}_{\text{Gr}_{B_n}}(N_{\geq s}, L).$$

There exist exact sequences: $0 \rightarrow M_{\geq k+1} \rightarrow M_{\geq k} \rightarrow M_{\geq k}/M_{\geq k+1} \rightarrow 0$ which induces an exact sequence:

$$0 \rightarrow \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}/M_{\geq k+1}, N) \rightarrow \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}, N) \rightarrow \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k+1}, N).$$

Since we are assuming N is torsion free $\text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}/M_{\geq k+1}, N) = 0$.

$$\text{Hence } \hat{f} \in \text{Hom}_{\text{Gr}_{B_n}}(\pi(M), \pi(N)) = \bigcup_{k \geq 0} \text{Hom}_{\text{Gr}_{B_n}}(M_{\geq k}, N).$$

The map \hat{f} is represented by $f: M_{\geq k} \rightarrow N$. Similarly, \hat{g} is represented by a map $g: N_{\geq \ell} \rightarrow L$ and we have a sequence: $M_{\geq k+\ell} \xrightarrow{f} N_{\geq \ell} \xrightarrow{g} L$ with $(\hat{g}\hat{f}) = \hat{g}\hat{f} = 0$, which implies gf factors through a torsion module, but L torsion free implies $gf = 0$. Since $M_{\geq k+\ell}$ is torsion free, f is a monomorphism. If Cokerg is torsion, there exists an $s \geq 0$ such that $\text{Cokerg}_{\geq s} = 0$. Taking a large enough truncation we obtain a sequence: $M_{\geq s} \xrightarrow{f} N_{\geq s} \xrightarrow{g} L_{\geq s}$ with f a monomorphism, g an epimorphism and $gf = 0$.

Consider the exact sequences: $0 \rightarrow M_{\geq s} \xrightarrow{f''} \text{Kerg} \rightarrow H \rightarrow 0$,

$$0 \rightarrow \text{Kerg} \rightarrow N_{\geq s} \rightarrow L_{\geq s} \rightarrow 0.$$

Applying π we obtain the following isomorphism of exact sequences:

$$\begin{array}{ccccccc} 0 \rightarrow & \pi(M_{\geq s}) & \xrightarrow{\pi f''} & \pi(\text{Kerg}) & \rightarrow & \pi(H) & \rightarrow 0 \\ & \downarrow \cong & & \downarrow \cong & & & \\ 0 \rightarrow & \pi(M) & \xrightarrow{\hat{f}} & \text{Kerg}^\wedge & \rightarrow & 0 & \end{array}$$

It follows $\pi(H) = 0$ and H is torsion, so there exists an integer $t \geq 0$ such that $H_{\geq t} = 0$. Finally taking a large enough truncation we get an exact sequence:

$0 \rightarrow M_{\geq s} \xrightarrow{f} N_{\geq s} \xrightarrow{g} L_{\geq s} \rightarrow 0$ such that the following sequences are isomorphic:

$$\begin{array}{ccccccc} 0 \rightarrow & \pi(M_{\geq s}) & \xrightarrow{\pi f} & \pi(N_{\geq s}) & \xrightarrow{\pi g} & \pi(L_{\geq s}) & \rightarrow 0 \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\ 0 \rightarrow & \pi(M) & \xrightarrow{\hat{f}} & \pi(N) & \xrightarrow{\hat{g}} & \pi(L) & \rightarrow 0 \end{array}$$

Applying ψ we have an exact sequence: $0 \rightarrow \psi\pi(M) \xrightarrow{\psi \hat{f}} \psi\pi(N) \xrightarrow{\psi \hat{g}} \psi\pi(L) \rightarrow 0$, which is isomorphic to $0 \rightarrow (M_{\geq s})_Z \xrightarrow{f_Z} (N_{\geq s})_Z \xrightarrow{g_Z} (L_{\geq s})_Z \rightarrow 0$.

We have proved ψ is exact. \square

The functor ψ has a derived functor: $D(\psi): D^b(\text{Qgr}_{B_n}) \rightarrow D^b(\text{gr}_{(B_n)_Z})$, we will study next its properties.

Observe Qgr_{B_n} does not have neither enough projective nor enough injective objects.

Lemma 1. *Let $0 \rightarrow N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \dots N_{\ell-1} \xrightarrow{d_{\ell-1}} N_\ell \rightarrow 0$ be a sequence of B_n -modules and assume the compositions $d_i d_{i-1}$ factors through a module of Z -torsion. Then there exists a complex:*

$$0 \rightarrow N_1 \xrightarrow{\hat{d}_1} N_2 \oplus t_Z(N_1) \xrightarrow{\hat{d}_2} N_3 \oplus t_Z(N_2) \dots N_{\ell-1} \oplus t_Z(N_{\ell-2}) \xrightarrow{\hat{d}_{\ell-1}} N_\ell \rightarrow 0, \text{ where}$$

$$\hat{d}_1 = \begin{bmatrix} -d_1 \\ s_1 \end{bmatrix}, \hat{d}_i = \begin{bmatrix} (-1)^i d_i & j_{i+1} \\ s_i & (-1)^i d'_{i+1} \end{bmatrix} \text{ and } d_{\ell-1}^\wedge = \begin{bmatrix} -d_{\ell-1} & j_\ell \end{bmatrix}, \text{ and the maps}$$

$j_i: t_Z(N_i) \rightarrow N_i$ are the natural inclusions.

Proof. Each morphism $d_i: N_i \rightarrow N_{i+1}$ induces by restriction a map $d'_i: t_Z(N_i) \rightarrow t_Z(N_{i+1})$ such that the following diagram commutes:

$$\begin{array}{ccccccc} t_Z(N_1) & \xrightarrow{d'_1} & t_Z(N_2) & \xrightarrow{d'_2} & t_Z(N_3) & \rightarrow \dots & t_Z(N_{\ell-1}) & \xrightarrow{d'_{\ell-1}} & t_Z(N_\ell) \\ \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 & & \downarrow j_{\ell-1} & & \downarrow j_\ell \\ N_1 & \xrightarrow{d_1} & N_2 & \xrightarrow{d_2} & N_3 & & N_{\ell-1} & \xrightarrow{d_{\ell-1}} & N_\ell \end{array}$$

Since the compositions $d_i d_{i-1}$ factors through a module of Z -torsion, there exist maps $s_i: N_i \rightarrow t_Z(N_{i+2})$ such that $j_{i+2} s_i = d_{i+1} d_i$.

We have the following equalities: $j_{i+2} s_i j_i = d_{i+1} d_i j_i = d_{i+1} j_i d'_i = j_{i+2} d'_{i+1} d'_i$ and j_{i+2} a monomorphism implies $s_i j_i = d'_{i+1} d'_i$.

We can easily check $d_{i+1}^\wedge d_i^\wedge = 0$. \square

Proposition 2. *Denote by Q the localization functor $Q = (B_n)_Z \otimes_B -$ and by $C^b(-)$, the category of bounded complexes. The induced functor $C^b(Q): C^b(\text{gr}_{B_n}) \rightarrow C^b(\text{gr}_{(B_n)_Z})$ is dense.*

Proof. Let $0 \rightarrow \hat{M}_1 \xrightarrow{\delta_1} \hat{M}_2 \xrightarrow{\delta_2} \dots \hat{M}_{\ell-1} \xrightarrow{\delta_{\ell-1}} \hat{M}_\ell \rightarrow 0$ be a complex in $C^b(\text{gr}_{(B_n)_Z})$.

For each \hat{M}_i there exists a finitely generated graded B_n -submodule M_i such that $(M_i)_Z \cong \hat{M}_i$ and a graded morphism $d_i: Z^{k_i} M_i \rightarrow M_{i+1}$ of B_n -modules such that $(d_i)_Z: (Z^{k_i} M_i)_Z \rightarrow (M_{i+1})_Z$ is isomorphic $\delta_i: \hat{M}_i \rightarrow \hat{M}_{i+1}$. Let k be $\sum_{i=0}^{\ell} k^i$. We then have a chain of B_n -morphisms:

$Z^k M_1 \xrightarrow{d_1} Z^{k_2 + \dots + k_\ell} M_2 \xrightarrow{d_2} Z^{k_3 + \dots + k_\ell} M_3 \xrightarrow{d_3} \dots Z^{k_{\ell-1} + \dots + k_\ell} M_{\ell-1} \xrightarrow{d_{\ell-1}} Z^{k_\ell} M_\ell$. Changing notation write M_i instead of $Z^{k_i + \dots + k_\ell} M_i$.

We then have a chain of morphisms:

$$M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \dots M_{\ell-1} \xrightarrow{d_{\ell-1}} M_\ell \text{ such that } (M_1)_Z \xrightarrow{d_1} (M_2)_Z \xrightarrow{d_2} \dots (M_{\ell-1})_Z \xrightarrow{d_{\ell-1}} (M_\ell)_Z$$

is isomorphic to the complex: $0 \rightarrow \hat{M}_1 \xrightarrow{\delta_1} \hat{M}_2 \xrightarrow{\delta_2} \dots \hat{M}_{\ell-1} \xrightarrow{\delta_{\ell-1}} \hat{M}_\ell \rightarrow 0$.

This implies $(d_i d_{i-1})_Z = 0$, which means $d_i d_{i-1}$ factors through a Z -torsion module. By lemma? there exists a complex:

$$0 \rightarrow M_1 \xrightarrow{\hat{d}_1} M_2 \oplus t_Z(M_1) \xrightarrow{\hat{d}_2} M_3 \oplus t_Z(M_2) \dots M_{\ell-1} \oplus t_Z(M_{\ell-1}) \xrightarrow{\hat{d}_{\ell-1}} M_\ell \rightarrow 0$$

such that

$$0 \rightarrow M_1 \xrightarrow{\hat{d}_1} (M_2 \oplus t_Z(M_1))_Z \xrightarrow{\hat{d}_2} (M_3 \oplus t_Z(M_2))_Z \dots (M_{\ell-1} \oplus t_Z(M_{\ell-1}))_Z \xrightarrow{\hat{d}_{\ell-1}} M_\ell \rightarrow 0$$

is isomorphic to: $0 \rightarrow \hat{M}_1 \xrightarrow{\delta_1} \hat{M}_2 \xrightarrow{\delta_2} \dots \hat{M}_{\ell-1} \xrightarrow{\delta_{\ell-1}} \hat{M}_\ell \rightarrow 0$. \square

Corollary 1. *The functor $C^b(\psi): C^b(Qgr_{B_n}) \rightarrow C^b(gr_{(B_n)_Z})$ is dense.*

Proof. There are functors $C^b(\pi): C^b(gr_{B_n}) \rightarrow C^b(Qgr_{B_n})$ and $C^b(\psi): C^b(Qgr_{B_n}) \rightarrow C^b(gr_{(B_n)_Z})$ such that $C^b(\psi) C^b(\pi) = C^b(Q)$ and $C^b(Q)$ dense implies $C^b(\psi)$ is dense. \square

Corollary 2. *The induced functors $K^b(Q): K^b(gr_{B_n}) \rightarrow K^b(gr_{(B_n)_Z})$ and $K^b(\psi): K^b(Qgr_{B_n}) \rightarrow K^b(gr_{(B_n)_Z})$ are dense.*

Proof. Interpreting $K^b(\mathcal{A})$ as the stable category of $C^b(\mathcal{A})$, denote by $\tau: C^b(\mathcal{A}) \rightarrow K^b(\mathcal{A})$ the corresponding functor. There is a commutative diagram:

$$\begin{array}{ccc} C^b(gr_{B_n}) & \xrightarrow{C^b(Q)} & C^b(gr_{(B_n)_Z}) \\ \downarrow \tau & & \downarrow \tau \\ K^b(gr_{B_n}) & \xrightarrow{K^b(Q)} & K^b(gr_{(B_n)_Z}) \end{array}$$

Since the functors τ and $C^b(Q)$ are dense, the functor $K^b(Q)$ is dense.

As above we have isomorphisms: $K^b(\psi) K^b(\pi) \cong K^b(Q)$. It follows $K^b(\psi)$ is dense. \square

Corollary 3. *The induced functors $D^b(Q): D^b(gr_{B_n}) \rightarrow D^b(gr_{(B_n)_Z})$ and $D^b(\psi): D^b(Qgr_{B_n}) \rightarrow D^b(gr_{(B_n)_Z})$ are dense.*

Proof. Since the functors $\pi: gr_{B_n} \rightarrow Qgr_{B_n}$ and $\psi: Qgr_{B_n} \rightarrow gr_{(B_n)_Z}$ are exact they induce derived functors $D^b(\pi): D^b(gr_{B_n}) \rightarrow D^b(Qgr_{B_n})$, $D^b(\psi): D^b(Qgr_{B_n}) \rightarrow D^b(gr_{(B_n)_Z})$ such that $D^b(\psi) D^b(\pi) = D^b(Q)$.

There is a commutative exact diagram:

$$\begin{array}{ccc} K^b(gr_{B_n}) & \xrightarrow{K^b(Q)} & K^b(gr_{(B_n)_Z}) \\ \downarrow & & \downarrow \\ D^b(gr_{B_n}) & \xrightarrow{D^b(Q)} & D^b(gr_{(B_n)_Z}) \end{array}$$

where the functors corresponding to the columns are dense, hence $D^b(Q)$ is dense, which in turn implies $D^b(\psi)$ is dense. \square

We will describe next the kernel of the functor $D^b(\psi)$. By definition, $\text{Ker } D^b(\psi) = \{\hat{M}^\circ \mid D^b(\psi)(\hat{M}^\circ) \text{ is acyclic}\}$.

Proposition 3. *There is the following description of $\mathcal{T} = \text{Ker } D^b(\psi)$. $\text{Ker } D^b(\psi) = \{\pi M^\circ \mid M^\circ \in D^b(gr_{B_n}) \text{ such that for all } i, H^i(M^\circ) \text{ is of } Z\text{-torsion}\}$.*

Proof. The kernel of the functor $D^b(\psi)$ is the category \mathcal{T} of complexes:

$$\tilde{N}^\circ : 0 \rightarrow \pi N_1 \xrightarrow{\hat{d}_1} \pi N_2 \xrightarrow{\hat{d}_2} \pi N_3 \dots \pi N_{\ell-1} \xrightarrow{\hat{d}_{\ell-1}} \pi N_\ell \rightarrow 0, \text{ such that:}$$

$$0 \rightarrow \psi \pi N_1 \xrightarrow{\psi \hat{d}_1} \psi \pi N_2 \xrightarrow{\psi \hat{d}_2} \psi \pi N_3 \dots \psi \pi N_{\ell-1} \xrightarrow{\psi \hat{d}_{\ell-1}} \psi \pi N_\ell \rightarrow 0 \text{ is acyclic.}$$

Proceeding as above, we may assume each N_i and each map \hat{d}_i lifts to a map $d_i : (N_i)_{\geq k} \rightarrow N_{i+1}$ such that the map $\pi(d_i) : \pi((N_i)_{\geq k}) \rightarrow \pi(N_{i+1})$ is isomorphic to $\hat{d}_i : \pi N_i \rightarrow \pi N_{i+1}$.

Taking a large enough truncation we get a complex of B_n -modules:

$$N_{\geq k}^\circ : 0 \rightarrow (N_1)_{\geq k} \xrightarrow{d_1} (N_2)_{\geq k} \xrightarrow{d_2} \dots (N_{\ell-1})_{\geq k} \xrightarrow{d_{\ell-1}} (N_\ell)_{\geq k} \rightarrow 0,$$

such that $\pi(N_{\geq k}^\circ) \cong \tilde{N}^\circ$.

The complex:

$$(N_{\geq k}^\circ)_Z : 0 \rightarrow (N_1)_{\geq k} \xrightarrow{d_1 Z} ((N_2)_{\geq k})_Z \xrightarrow{d_2 Z} \dots ((N_{\ell-1})_{\geq k})_Z \xrightarrow{d_{\ell-1} Z} ((N_\ell)_{\geq k})_Z \rightarrow 0 \text{ is}$$

isomorphic to $0 \rightarrow \psi \pi N_1 \xrightarrow{\psi \hat{d}_1} \psi \pi N_2 \xrightarrow{\psi \hat{d}_2} \psi \pi N_3 \dots \psi \pi N_{\ell-1} \xrightarrow{\psi \hat{d}_{\ell-1}} \psi \pi N_\ell \rightarrow 0$, hence it is acyclic.

Changing notation we have a complex of B_n -modules:

$$N^\circ : 0 \rightarrow N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \dots N_{\ell-1} \xrightarrow{d_{\ell-1}} N_\ell \rightarrow 0 \text{ such that}$$

$$(N^\circ)_Z : 0 \rightarrow (N_1)_Z \xrightarrow{d_1 Z} (N_2)_Z \xrightarrow{d_2 Z} \dots (N_{\ell-1})_Z \xrightarrow{d_{\ell-1} Z} (N_\ell)_Z \rightarrow 0 \text{ is acyclic.}$$

We have exact sequences:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ 0 \rightarrow & \text{Kerd}_1 & \rightarrow & N_1 & \xrightarrow{d_1} & \text{Imd}_1 & \rightarrow 0 \\ & & & \downarrow j & & & \\ & 0 \rightarrow & \text{Kerd}_2 & \rightarrow & N_2 & \xrightarrow{d_2} & N_3 \\ & & \downarrow & & & & \\ & & H^1(N^\circ) & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

Localizing we get exact sequences:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ 0 \rightarrow & (\text{Kerd}_1)_Z & \rightarrow & (N_1)_Z & \xrightarrow{d_1 Z} & (\text{Imd}_1)_Z & \rightarrow 0 \\ & & & \downarrow j_Z & & & \\ & 0 \rightarrow & (\text{Kerd}_2)_Z & \rightarrow & (N_2)_Z & \xrightarrow{d_2 Z} & (N_3)_Z \\ & & \downarrow & & & & \\ & & H^1(N^\circ)_Z & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where j_Z, d_{1Z} are isomorphisms, hence $H^0(N^\circ)_Z = 0, H^1(N^\circ)_Z = 0$. Therefore $H^0(N^\circ)$ and $H^1(N^\circ)$ are \mathbb{Z} -torsion. More generally for each i the modules $H^i(N^\circ)$ are \mathbb{Z} -torsion. \square

Remark 1. Let M be a Koszul left $B^!$ -module such that $F(M) = \bigoplus_{n \geq 0} \text{Ext}_{B^!}^n(M, B_0^!)$ is a B -module of Z -torsion and $\sigma : B_n \rightarrow B_n$ the Nakayama automorphism defined in Section 1. Then $F(\sigma M)$ is of Z -torsion, in particular $F(D(M^*)[n])$ is of Z -torsion.

Since $F(\sigma M) = \sigma FM$ for $x \in F(M)$ there is an integer $k \geq 0$ such that $Z^k x = 0$ and in σFM , $Z^k * x = \sigma(Z^k)_x = c^k Z^k x = 0$.

Corollary 4. The Nakayama automorphism $\sigma : B_n \rightarrow B_n$ induces an autoequivalence $D^b(\sigma) : D^b(\text{gr}_{B_n}) \rightarrow D^b(\text{gr}_{B_n})$ and \mathcal{T} is invariant under $D^b(\sigma)$.

Proof. We saw in Section 1 that given an automorphism of graded algebras $\sigma : B_n \rightarrow B_n$, there is an autoequivalence $\text{gr}_{B_n} \rightarrow \text{gr}_{B_n}$, that we also denote by σ , such that $\sigma(M)$ is the module M with twisted multiplication $b \in B_n$ and $m \in M$, $b * m = \sigma(b)m$, clearly σ is an exact functor that sends modules of finite length into modules of finite length. Then σ induces an exact functor: $\sigma : \text{Qgr}_{B_n} \rightarrow \text{Qgr}_{B_n}$. Therefore an autoequivalence: $D^b(\sigma) : D^b(\text{gr}_{B_n}) \rightarrow D^b(\text{gr}_{B_n})$. If M is a module of Z -torsion, then σM is of Z -torsion. From this it is clear that $D^b(\sigma)$ sends an element of \mathcal{T} to an element of \mathcal{T} . \square

The category $\mathcal{T} = \text{Ker} D(\psi)$ is "epasse" (thick) and we can take the Verdier quotient $D^b(\text{Qgr}_{B_n}) / \mathcal{T}$. [Mi]

Our aim is to prove the main result of the section:

Theorem 1. There exists an equivalence of triangulated categories:

$$D^b(\text{Qgr}_{B_n}) / \mathcal{T} \cong D^b(\text{gr}_{(B_n)_Z}).$$

Let $\hat{s}^{-1} \hat{f} : \hat{X}^\circ \rightarrow \hat{Y}^\circ$ be a map in $\Psi(\mathcal{T})$ (in Miyachi's notation). This is a roof

$$\begin{array}{ccc} & \hat{K}^\circ & \\ \hat{f} \nearrow & & \nwarrow \hat{s} \\ \hat{X}^\circ & & \hat{Y}^\circ \end{array},$$

where \hat{X}° is a complex of the form:

$$\hat{X}^\circ : 0 \rightarrow \pi X^{n_0} \xrightarrow{\hat{d}} \pi X^{n_0+1} \xrightarrow{\hat{d}} \dots \rightarrow \pi X^{n_0+\ell-1} \xrightarrow{\hat{d}} \pi X^{n_0+\ell} \rightarrow 0.$$

After a proper truncation there exists complexes of graded B_n -modules X° , K° , Y° such that $\pi X^\circ \cong \hat{X}^\circ$, $\pi K^\circ \cong \hat{K}^\circ$, $\pi Y^\circ \cong \hat{Y}^\circ$ and graded maps $f : X^\circ \rightarrow K^\circ$, $s : Y^\circ \rightarrow K^\circ$ such that $\pi f = \hat{f}$, $\pi s = \hat{s}$, the roof $\hat{s}^{-1} \hat{f}$ becomes:

$$\begin{array}{ccc} & \pi K^\circ & \\ \pi f \nearrow & & \nwarrow \pi s \\ \pi X^\circ & & \pi Y^\circ \end{array},$$

where πs is a quasi isomorphism.

There is a triangle in $K^b(\text{gr}_{B_n})$: $X^\circ \xrightarrow{f} K^\circ \xrightarrow{g} Z^\circ \xrightarrow{h} X^\circ[-1]$ which induces a morphism of triangles:

$$\begin{array}{ccccccc} X^\circ & \xrightarrow{f} & K^\circ & \xrightarrow{g} & Z^\circ & \xrightarrow{h} & X^\circ[-1] \\ \uparrow u & & \uparrow s & & \uparrow 1 & & \uparrow u[-1] \\ X'^\circ & \rightarrow & Y^\circ & \xrightarrow{gs} & Z^\circ & \rightarrow & X'^\circ[-1] \end{array}$$

Applying π we obtain a morphism of triangles:

$$\begin{array}{ccccccc} \pi X^\circ & \xrightarrow{\pi f} & \pi K^\circ & \xrightarrow{\pi g} & \pi Z^\circ & \xrightarrow{\pi h} & \pi X^\circ[-1] \\ \uparrow \pi u & & \uparrow \pi s & & \uparrow 1 & & \uparrow \pi u[-1] \\ \pi X'^\circ & \rightarrow & \pi Y^\circ & \xrightarrow{\pi g s} & \pi Z^\circ & \rightarrow & \pi X'^\circ[-1] \end{array}$$

By definition of $\Psi(\mathcal{T})$ the object $\pi Z^\circ \in \mathcal{T}$, which means Z° has homology of Z -torsion. The maps $\pi s, \pi u$ are quasi isomorphisms. Applying the functor ψ we obtain a triangle: $\psi \pi X^\circ \xrightarrow{\psi \pi f} \psi \pi K^\circ \xrightarrow{\psi \pi g} \psi \pi Z^\circ \xrightarrow{\psi \pi h} \psi \pi X^\circ[-1]$ where $\psi \pi Z^\circ$ is acyclic. It follows $\psi \pi f$ is invertible in $D^b(\text{gr}_{(B_n)_Z})$.

We have proved the functor: $D^b(\psi): D^b(\text{Qgr}_{B_n}) \rightarrow D^b(\text{gr}_{(B_n)_Z})$ sends elements of $\Psi(\mathcal{T})$ to invertible elements in $D^b(\text{gr}_{(B_n)_Z})$. By [Mi] Prop. 712, there exists a functor $\theta: D^b(\text{Qgr}_{B_n})/\mathcal{T} \rightarrow D^b(\text{gr}_{(B_n)_Z})$ such that the triangle:

$$\begin{array}{ccc} D^b(\text{Qgr}_{B_n}) & \xrightarrow{D^b(\psi)} & D^b(\text{gr}_{(B_n)_Z}) \\ & \searrow \theta & \nearrow \theta \\ & D^b(\text{Qgr}_{B_n})/\mathcal{T} & \end{array},$$

commutes.

Since $D^b(\psi)$ is dense, so is θ .

Before proving θ is an equivalence, we will need two lemmas:

Lemma 2. *Let K°, L° be complexes in $C^b(\text{gr}_{B_n})$ and let $\hat{f}: K_Z^\circ \rightarrow L_Z^\circ$ be a morphism of complexes of graded $(B_n)_Z$ -modules. Then there exists a bounded complex of graded B_n -modules N° and a map of complexes $f: N^\circ \rightarrow L^\circ$ such that $N_Z^\circ \cong K_Z^\circ$ and $f_Z = \hat{f}$.*

Proof. Let K°, L° be the complexes: $K^\circ: 0 \rightarrow K^0 \xrightarrow{d} K^1 \xrightarrow{d} \dots K^{n-1} \xrightarrow{d} K^n \rightarrow 0$ and $L^\circ: 0 \rightarrow L^0 \xrightarrow{d} L^1 \xrightarrow{d} \dots L^{n-1} \xrightarrow{d} L^n \rightarrow 0$.

Each map $\hat{f}_i: K_Z^i \rightarrow L_Z^i$ lifts to a map $f_i: Z^{k_i} K^i \rightarrow L^i$ such that $(f_i)_Z = \hat{f}_i$. Let $k = \max\{k_j\}$. Then we have the following diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & Z^k K^0 & \xrightarrow{d} & Z^k K^1 & \xrightarrow{d} & Z^k K^2 & \xrightarrow{d} \dots Z^k K^n \rightarrow 0 \\ & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_n \\ 0 \rightarrow & L^0 & \xrightarrow{d} & L^1 & \xrightarrow{d} & L^2 & \xrightarrow{d} \dots L^n \rightarrow 0 \end{array}$$

where $(df_{i-1} - f_i d)_Z = d \hat{f}_{i-1} - \hat{f}_i d = 0$. The map $df_{i-1} - f_i d$ factors through $t_Z(L^i)$.

There exist maps $s_{i-1}: Z^k K^{i-1} \rightarrow t_Z(L^i)$, $j_i: t_Z(L^i) \rightarrow L^i$ such that $f_i d - df_{i-1} = j_i s_{i-1}$ and the diagrams:

$$\begin{array}{ccc} t_Z(L^{i-1}) & \xrightarrow{d'} & t_Z(L^i) \\ \downarrow j_{i-1} & & \downarrow j_i \\ L^{i-1} & \xrightarrow{d} & L^i \end{array}$$

commute.

We have the following equalities: $(f_i d - df_{i-1})d = j_i s_{i-1} d$, $-df_{i-1}d = j_i s_{i-1} d$ and $d(f_{i-1}d - df_{i-2}) = df_{i-1}f = dj_{i-1}s_{i-2} = j_i ds_{i-2}$.

But j_i mono implies $s_{i-1}d + ds_{i-2} = 0$.

We have proved that the sequence

$$N^\circ: 0 \rightarrow Z^k K^0 \xrightarrow{d_0} Z^k K^1 \oplus t_Z(L^1) \xrightarrow{d_1} Z^k K^2 \oplus t_Z(L^2) \dots Z^k K^{n-1} \oplus t_Z(L^{n-1}) \xrightarrow{d_{n-1}} Z^k K^n \rightarrow 0,$$

with maps: $\hat{d}_0 = \begin{bmatrix} d \\ s_0 \end{bmatrix}$, $\hat{d}_i = \begin{bmatrix} d & 0 \\ s_i & d' \end{bmatrix}$, $\hat{d}_{\ell-1} = \begin{bmatrix} d & 0 \end{bmatrix}$ is a complex of B_n -modules and $(f_i, -j_i): Z^k K^i \oplus t_Z(L^i) \rightarrow L^i$, $(f, -j): N^\circ \rightarrow L^\circ$ is a map of complexes such that $N_Z^\circ \cong K_Z^\circ$ and $(f, j)_Z = \hat{f}$. \square

Lemma 3. *Let K°, L° be complexes in $C^b(\text{gr}_{B_n})$ and $\hat{f}: L_Z^\circ \rightarrow K_Z^\circ$ be a morphism of complexes of graded $(B_n)_Z$ -modules which is homotopic to zero. Then there exist bounded complexes of B_n -modules, M°, N° and a map of complexes $f: N^\circ \rightarrow M^\circ$, such that f is homotopic to zero, $N_Z^\circ \cong L_Z^\circ$, $M_Z^\circ \cong K_Z^\circ$ and $f_Z \cong \hat{f}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccccccc} 0 \rightarrow & L_Z^0 & \xrightarrow{d_Z} & L_Z^1 & \xrightarrow{d_Z} & L_Z^2 & \xrightarrow{d_Z} \dots & L_Z^m & \rightarrow 0 \\ & \downarrow \hat{f}_0 & s_1 \swarrow & \downarrow \hat{f}_1 & s_2 \swarrow & \downarrow \hat{f}_2 & s_m \swarrow & \downarrow \hat{f}_m & \\ 0 \rightarrow & K_Z^0 & \xrightarrow{d_Z} & K_Z^1 & \xrightarrow{d_Z} & K_Z^2 & \xrightarrow{d_Z} \dots & K_Z^n & \rightarrow 0 \end{array},$$

where $s: L_Z^\circ \rightarrow K_Z^\circ[-1]$ is the homotopy, hence $\hat{f}_i = d_Z s_i + s_{i+1} d_Z$.

For each i there exist integers k_i, k'_i and maps $t_i: Z^{k_i} L^i \rightarrow K^{i-1}$ and $f'_i: Z^{k'_i} L^i \rightarrow K^i$ such that $(t_i)_Z = s_i$ and $(f'_i)_Z = \hat{f}_i$. Taking $k = \max\{k_j\}$ we have maps:

$$\begin{array}{ccccccccccc} 0 \rightarrow & Z^k L^0 & \xrightarrow{d} & Z^k L^1 & \xrightarrow{d} & Z^k L^2 & \xrightarrow{d} \dots & Z^k L^m & \rightarrow 0 \\ & \downarrow f'_0 & t_1 \swarrow & \downarrow f'_1 & t_2 \swarrow & \downarrow f'_2 & t_m \swarrow & \downarrow f'_m & \\ 0 \rightarrow & K^0 & \xrightarrow{d} & K^1 & \xrightarrow{d} & K^2 & \xrightarrow{d} \dots & K^m & \rightarrow 0 \end{array}$$

Consider the map: $(f'_i - (t_{i+1} d + dt_i))_Z = \hat{f}_i - (s_{i+1} d_Z + d_Z s_i) = 0$. As above, $f'_i - (t_{i+1} d + dt_i)$ factors through a Z -torsion module and there exist maps: $v_i: Z^k L^i \rightarrow t_Z(K^i)$ and inclusions $j_i: t_Z(K^i) \rightarrow K^i$ such that $f'_i - (t_{i+1} d + dt_i) = -j_i v_i$ or $f'_i + j_i v_i = t_{i+1} d + dt_i$.

Set $f_i = f'_i + j_i v_i$. Then $(f_i)_Z = (f'_i)_Z = \hat{f}_i$.

But we have now $f_i = t_{i+1} d + dt_i, f_{i-1} = t_i d + dt_{i-1}$ imply $f_i d = dt_i d = f_{i-1} d$. \square

We can prove now the theorem.

i) θ is faithful.

Let \mathcal{T}_1 be $\mathcal{T}_1 = \{X^\circ \in C^b(\text{gr}_{B_n}) \mid H^i(X^\circ) \text{ is torsion for all } i\}$ and $\mathcal{T}_2 = \{X^\circ \in C^b(\text{gr}_{B_n}) \mid H^i(X^\circ) \text{ is } Z\text{-torsion for all } i\}$.

A map in $D^b(\text{Qgr}_{B_n})/\mathcal{T}$ can be written as follows:

$$\begin{array}{ccccccc} & & \pi K^\circ & & \pi L^\circ & & \\ & \nearrow \pi f & \nwarrow \pi s & \nearrow \pi t & \nwarrow \pi g & & \\ \pi X^\circ & & \pi Y^\circ & & \pi Z^\circ & & \end{array}$$

where $t, s \in \Psi(\mathcal{T}_1)$ and $g \in \Psi(\mathcal{T}_2)$.

In $K^b(\text{gr}_{B_n})$ we have maps:

$$\begin{array}{ccccc} & & K^\circ & & L^\circ \\ & \nearrow f & \nwarrow s & \nearrow t & \nwarrow g \\ X^\circ & & Y^\circ & & Z^\circ \end{array}$$

We have an exact sequence of complexes:

$$0 \rightarrow Y^\circ \xrightarrow{\mu} K^\circ \oplus L^\circ \oplus I^\circ \xrightarrow{\nu} W^\circ \rightarrow 0$$

Where I° is a complex which is a sum of complexes of the form: $0 \rightarrow X \xrightarrow{1} X \rightarrow 0$, hence acyclic. The maps μ, v are of the form: $\mu = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$ and $v = \begin{bmatrix} t' & s' & v \end{bmatrix}$.

By the long homology sequence, there is an exact sequence: *)

$$\dots \rightarrow H^{i+1}(W^\circ) \rightarrow H^i(Y^\circ) \xrightarrow{H^i(\mu)} H^i(K^\circ) \oplus H^i(L^\circ) \xrightarrow{H^i(v)} H^i(W^\circ) \rightarrow H^{i-1}(Y^\circ) \rightarrow \dots$$

Since π is an exact functor, for any complex $\pi H^i(X^\circ) \cong H^i(\pi X^\circ)$ and the exact sequence *) induces an exact sequence: **)

$$\dots \rightarrow \pi H^{i+1}(W^\circ) \rightarrow \pi H^i(Y^\circ) \xrightarrow{\pi H^i(\mu)} \pi H^i(K^\circ) \oplus \pi H^i(L^\circ) \xrightarrow{\pi H^i(v)} \pi H^i(W^\circ) \rightarrow \pi H^{i-1}(Y^\circ) \rightarrow \dots$$

Which is isomorphic to the complex:

$$\dots \rightarrow H^{i+1}(\pi W^\circ) \rightarrow H^i(\pi Y^\circ) \xrightarrow{H^i(\pi \mu)} H^i(\pi K^\circ) \oplus H^i(\pi L^\circ) \xrightarrow{H^i(\pi v)} H^i(\pi W^\circ) \rightarrow H^{i-1}(\pi Y^\circ) \rightarrow \dots$$

The maps $H^i(\pi s)$, $H^i(\pi t)$ are isomorphisms. Hence it follows $H^i(\pi \mu)$ is for each i a splittable monomorphism and for each i there is an exact sequence:

$$0 \rightarrow H^i(\pi Y^\circ) \xrightarrow{H^i(\pi \mu)} H^i(\pi K^\circ) \oplus H^i(\pi L^\circ) \xrightarrow{H^i(\pi v)} H^i(\pi W^\circ) \rightarrow 0$$

which can be embedded in a commutative exact diagram:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & H^i(\pi L^\circ) & & \xrightarrow{1} & H^i(\pi L^\circ) \\ & & & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & & \downarrow H^i(\pi s') \\ 0 \rightarrow & H^i(\pi Y^\circ) & \rightarrow & H^i(\pi K^\circ) \oplus H^i(\pi L^\circ) & \rightarrow & H^i(\pi W^\circ) & \rightarrow 0 \\ & \downarrow H^i(\pi s) & & \downarrow \begin{bmatrix} 1 & 0 \end{bmatrix} & & \downarrow & \\ 0 \rightarrow & H^i(\pi K^\circ) & \xrightarrow{1} & H^i(\pi K^\circ) & \rightarrow & 0 & \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

By this and a similar diagram it follows $H^i(\pi s')$, $H^i(\pi t')$ are isomorphisms.

We have a commutative diagram in $K^b(\text{Qgr}_{B_n})$:

$$\begin{array}{ccccccc} & & & \pi W^\circ & & & \\ & & \nearrow \pi t' & & \nwarrow \pi s' & & \\ & \pi K^\circ & & & & \pi L^\circ & \\ \nearrow \pi f & & \nwarrow \pi s & & \nearrow \pi t & & \nwarrow \pi g \\ \pi X^\circ & & & \pi Y^\circ & & & \pi Z^\circ \end{array}$$

Then $\theta((\pi g)^{-1} \pi t (\pi s)^{-1} \pi f) = D^b(\psi)((\pi s' \pi g)^{-1} \pi t' \pi f) = (s'_Z g_Z)^{-1} t'_Z f_Z = 0$.

But s'_Z , g_Z , t'_Z are isomorphisms in $D^b(\text{gr}_{B_z})$. It follows $f_Z = 0$ in $D^b(\text{gr}_{B_z})$.

Then there is a quasi isomorphism of complexes $v : \hat{N}^\circ \rightarrow X_Z^\circ$ such that $f_Z v$ is homotopic to zero. By Lemma 3, there is a bounded complex N° of B_n -modules and a map $\nu : N^\circ \rightarrow X^\circ$ such that $N_Z^\circ \cong \hat{N}^\circ$ and ν_Z can be identified with v .

According to Lemma 3. there is an integer $k \geq 0$ such that the composition of maps $Z^k N^\circ \xrightarrow{res\nu} X^\circ \xrightarrow{f} K^\circ$ is homotopic to zero and $(res\nu)_Z = \nu_Z$ is a quasi isomorphism. This implies $res\nu \in \Psi(\mathcal{T}_2)$ and $\pi f = 0$ in $D^b(\text{Qgr}_{B_n})/\mathcal{T}$.

Therefore $\pi g)^{-1} \pi t(\pi s)^{-1} \pi f = 0$ in $D^b(\text{Qgr}_{B_n})/\mathcal{T}$.

ii) θ is full.

Let

$$\begin{array}{ccc} & K_Z^\circ & \\ \hat{s} \swarrow & & \searrow \hat{f} \\ X_Z^\circ & & Y_Z^\circ \end{array}$$

be a map in $D^b(\text{gr}_{(B_n)_Z})$. By lemma 1, there exists a complex:

$$N^\circ: 0 \rightarrow Z^k K^0 \xrightarrow{\hat{d}_0} Z^k K^1 \oplus t_Z(Y^1) \xrightarrow{\hat{d}_1} Z^k K^2 \oplus t_Z(Y^2) \dots Z^k K^{n-1} \oplus t_Z(Y^{n-1}) \xrightarrow{\hat{d}_{\ell-1}} Z^k K^n \rightarrow 0,$$

where the differentials are of the form: $\hat{d}_0 = \begin{bmatrix} d \\ s_0 \end{bmatrix}$, $\hat{d}_i = \begin{bmatrix} d & 0 \\ s_i & d' \end{bmatrix}$, $\hat{d}_{\ell-1} = \begin{bmatrix} d & 0 \end{bmatrix}$

and a map $f: N^\circ \rightarrow Y^\circ$ such that $N_Z^\circ \cong K_Z^\circ$ and $f_Z = \hat{f}$. Changing N° for K° we may assume \hat{f} is a localized map f_Z and get a roof:

$$\begin{array}{ccc} & N_Z^\circ & \\ \hat{s} \swarrow & & \searrow f_Z \\ X_Z^\circ & & Y_Z^\circ \end{array}.$$

We now lift \hat{s} to a map of complexes $s: \hat{N}^\circ \rightarrow X^\circ$:

$$\begin{array}{ccccccc} 0 \rightarrow & Z^k N^0 & \xrightarrow{\hat{d}_0} & Z^k N^1 \oplus t_Z(X^1) & \xrightarrow{\hat{d}_1} & Z^k N^2 \oplus t_Z(X^2) & \dots \xrightarrow{\hat{d}_{m-1}} Z^k N^m \rightarrow 0 \\ s: & \downarrow s_0 & & \downarrow s_1 & & \downarrow s_2 & & \downarrow s_m \\ 0 \rightarrow & X^0 & \xrightarrow{d} & X^1 & \xrightarrow{d} & X^2 & \dots \xrightarrow{d} & X^m \rightarrow 0 \end{array}$$

with $s_z = \hat{s}$.

We have a commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & Z^k N^0 & \xrightarrow{\hat{d}_0} & Z^k N^1 \oplus t_Z(X^1) & \xrightarrow{\hat{d}_1} & Z^k N^2 \oplus t_Z(X^2) & \dots \xrightarrow{\hat{d}_{m-1}} Z^k N^m \rightarrow 0 \\ (10) & \downarrow 1 & & \downarrow (10) & & \downarrow (10) & & \downarrow 1 \\ 0 \rightarrow & Z^k N^0 & \xrightarrow{d} & Z^k N^1 & \xrightarrow{d} & Z^k N^2 & \dots \xrightarrow{d} & Z^k N^m \rightarrow 0 \end{array}$$

We obtain the following roof:

$$\begin{array}{ccc} & \hat{N}^\circ & \\ s \swarrow & & \searrow (10) \\ X^\circ & & Z^k N^\circ \\ & & \searrow f \\ & & Y^\circ \end{array}$$

Localizing we obtain:

$$\begin{array}{ccc} & \hat{N}_Z^\circ & \\ s_Z \swarrow & & \searrow (10)_Z \\ X_Z^\circ & & Z^k N_Z^\circ \\ & & \searrow f_Z \\ & & Y_Z^\circ \end{array}$$

with $N_Z^{\circ} \xrightarrow{(10)} Z^k N_Z^{\circ} \cong N_Z^{\circ}$ isomorphisms, $s_Z = \hat{s}$, $f_Z = \hat{f}$.
 We have proved θ is full.

3. THE CATEGORY OF \mathcal{T} -LOCAL OBJECTS.

Let \mathcal{F} be the full subcategory of $D^b(\text{Qgr}_{B_n})$ consisting of \mathcal{T} -local objects, this is: $\mathcal{F} = \{X^{\circ} \in D^b(\text{Qgr}_{B_n}) | \text{Hom}_{D^b(\text{Qgr}_{B_n})}(\mathcal{T}, X^{\circ}) = 0\}$.

According to [Mi], Prop. 9.8, for each $Y^{\circ} \in D^b(\text{Qgr}_{B_n})$ and $X^{\circ} \in \mathcal{F}$, $\text{Hom}_{D^b(\text{Qgr}_{B_n})}(Y^{\circ}, X^{\circ}) = \text{Hom}_{D^b(\text{Qgr}_{B_n})/\mathcal{T}}(QY^{\circ}, QX^{\circ}) \cong \text{Hom}_{D^b(\text{gr}_{(B_n)_Z})}(\psi Y^{\circ}, \psi X^{\circ})$.

In particular there is a full embedding of \mathcal{F} in $D^b(\text{gr}_{(B_n)_Z})$.

According to [MM] and [MS] there is a duality of triangulated categories $\bar{\phi} : \underline{\text{gr}}_{B_n^{\text{!op}}} \rightarrow D^b(\text{Qgr}_{B_n})$ induced by the duality $\phi : \text{gr}_{B_n^{\text{!op}}} \rightarrow \mathcal{LCP}_{B_n}$, with \mathcal{LCP}_{B_n} the category of linear complexes of graded projective B_n -modules. If $M = \bigoplus_{i \geq k_0} M_i$

is a graded $B_n^{\text{!op}}$ -module, then $\phi(M)$ is a complex of the form:

$$\begin{aligned} D(M) \otimes_{B_0} B_n &: \rightarrow \dots \rightarrow D(M_{k_0+n}) \otimes_{B_0} B_n[-k_0-n] \rightarrow D(M_{k_0+n-1}) \otimes_{B_0} B_n[-k_0-n+1] \rightarrow \dots \\ D(M_{k_0+1}) \otimes_{B_0} B_n[-k_0-1] &\rightarrow D(M_{k_0}) \otimes_{B_0} B_n[-k_0] \rightarrow 0. \end{aligned}$$

$\bar{\phi}(M)$ is the complex:

$$\begin{aligned} \pi(D(M) \otimes_{B_0} B_n) &: \rightarrow \dots \rightarrow \pi(D(M_{k_0+n}) \otimes_{B_0} B_n[-k_0-n]) \rightarrow \pi(D(M_{k_0+n-1}) \otimes_{B_0} B_n[-k_0-n+1]) \\ \rightarrow \dots \rightarrow \pi(D(M_{k_0+1}) \otimes_{B_0} B_n[-k_0-1]) &\rightarrow \pi(D(M_{k_0}) \otimes_{B_0} B_n[-k_0]) \rightarrow 0. \end{aligned}$$

If we compose with the usual duality we obtain an equivalence of triangulated categories: $\bar{\phi}D : \underline{\text{gr}}_{B_n^{\text{!op}}} \rightarrow D^b(\text{Qgr}_{B_n})$.

Under the duality $\bar{\phi}$ there is a pair $(\mathcal{F}', \mathcal{T}')$ such that $\mathcal{F}' \rightarrow \mathcal{F}$ and $\mathcal{T}' \rightarrow \mathcal{T}$ corresponds to the pair $(\mathcal{T}, \mathcal{F})$.

We want to characterize the subcategories $\mathcal{F}', \mathcal{T}'$ of $\underline{\text{gr}}_{B_n^{\text{!op}}}$.

We shall start by recalling some properties of the finitely generated graded B_n -modules.

The algebra B_n is a Koszul algebra of finite global dimension, under such conditions, for any finitely generated graded B_n -module M there is a truncation $M_{\geq k}$ such that $M_{\geq k}[k]$ is Koszul [M2]. But in Qgr_{B_n} the objects πM and $\pi M_{\geq k}$ are isomorphic, hence we can consider only Koszul B_n -modules and their shifts. Assume M is finitely generated but of infinite dimension over \mathbb{k} . The torsion part $t(M)$ is finite dimensional over \mathbb{k} , hence there is a torsion free truncation $M_{\geq k}$ of M , so we may assume M torsion free and Koszul.

Let's suppose M is of Z -torsion. There exists an integer n such that $Z^{n-1}M \neq 0$ and $Z^n M = 0$. There is a filtration $M \supset ZM \supset Z^2M \dots \supset Z^{n-1}M \supset 0$. Since Z is an element of degree one $(ZM)_i = ZM_{i-1}$, which implies $(Z^j M)_{\geq k} = Z^j(M_{\geq k-j})$.

Truncation of Koszul is Koszul and we can take large enough truncation in order to have $(Z^j M)_{\geq k}$ Koszul for all j . Changing M for $M_{\geq k}$ we may assume all $Z^j M$ are Koszul. [GM1], [GM2].

There is a commutative exact diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & \Omega(M) & \rightarrow & \Omega(M/ZM) & \rightarrow & ZM & \rightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& P & \xrightarrow{1} & P & & & \\
& \downarrow & & \downarrow & & & \\
0 \rightarrow & ZM & \rightarrow & M & \rightarrow & M/ZM & \rightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

the modules $\Omega(M)$, ZM are Koszul generated in the same degree, it follows M/ZM is Koszul and for any integer $k \geq 1$ there is an exact sequence:

$0 \rightarrow \Omega^k(M) \rightarrow \Omega^k(M/ZM) \rightarrow \Omega^{k-1}(ZM) \rightarrow 0$. By [GM1] there is an exact sequence:

$$0 \rightarrow \text{Hom}_{B_n}(\Omega^{k-1}(ZM), B_{n0}) \rightarrow \text{Hom}_{B_n}(\Omega^k(M/ZM), B_{n0}) \rightarrow$$

$$\text{Hom}_{B_n}(\Omega^k(M), B_{n0}) \rightarrow 0 \text{ or an exact sequence:}$$

$$*) 0 \rightarrow \text{Ext}_{B_n}^{k-1}(ZM, B_{n0}) \rightarrow \text{Ext}_{B_n}^k(M/ZM, B_{n0}) \rightarrow \text{Ext}_{B_n}^k(M, B_{n0}) \rightarrow 0.$$

We will denote by $F_{B_n}(N) = \bigoplus_{k \geq 0} \text{Ext}_{B_n}^k(N, B_{n0})$ the Koszul duality functor

$$F_{B_n}: K_{B_n} \rightarrow K_{B_n}^!.$$

Adding all sequences *) we obtain an exact sequence:

$$0 \rightarrow F_{B_n}(ZM)[-1] \rightarrow F_{B_n}(M/ZM) \rightarrow F_{B_n}(M) \rightarrow 0$$

We can apply the same argument to any module $Z^j M$ to get an exact sequence:

$$0 \rightarrow F_{B_n}(Z^{j+1}M)[-j-1] \rightarrow F_{B_n}(Z^j M/Z^{j+1}M)[-j] \rightarrow F_{B_n}(Z^j M)[-j] \rightarrow 0.$$

Gluing all short exact sequences we obtain a long exact sequence of Koszul up to shifting $B_n^!$ -modules:

$$**) 0 \rightarrow F_{B_n}(Z^{n-1}M)[-n+1] \rightarrow F_{B_n}(Z^{n-2}M/Z^{n-1}M)[-n+2] \rightarrow \dots \rightarrow$$

$$F_{B_n}(M/ZM) \rightarrow F_{B_n}(M) \rightarrow 0$$

It will be enough to study non semisimple Koszul B_n -modules N such that $ZN = 0$. They can be considered as C_n -modules.

We have the following commutative exact diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & ZB_n^{n_0} & \xrightarrow{1} & ZB_n^{n_0} & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & \Omega_B(N) & \rightarrow & B_n^{n_0} & \rightarrow & N & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow 1 & \\
0 \rightarrow & \Omega_C(N) & \rightarrow & C_n^{n_0} & \rightarrow & N & \rightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

The algebra B_n is an integral domain and in consequence the free B_n -modules are torsion free and $ZB_n^{n_0}$ is isomorphic to $B_n^{n_0}[-1]$.

The exact sequence: $0 \rightarrow ZB_n^{n_0} \rightarrow \Omega_B(N) \rightarrow \Omega_C(N) \rightarrow 0$ consists of graded modules generated in degree one and the first two term are Koszul, by [GM] this implies $\Omega_C(N)$ is Koszul as B_n -module.

There is a commutative exact diagram:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
& & 0 & \rightarrow & B_n^{n_0}[-1] & \rightarrow & ZB_n^{n_0} \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & \Omega_B^2(N) & \rightarrow & B_n^{n_0+n_1}[-1] & \rightarrow & \Omega_B(N) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \Omega_B \Omega_C(N) & \rightarrow & B_n^{n_1}[-1] & \rightarrow & \Omega_C(N) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

In particular $\Omega_B^2(N) \cong \Omega_B \Omega_C(N)$.

Since $\Omega_C(N) \subset C_n^{n_0}$ it is a C_n -module and we have the following commutative exact diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & ZB_n^{n_1}[-1] & \xrightarrow{1} & ZB_n^{n_1}[-1] & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & \Omega_B \Omega_C(N) & \rightarrow & B_n^{n_1}[-1] & \rightarrow & \Omega_C(N) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow 1 & \\
0 \rightarrow & \Omega_C^2(N) & \rightarrow & C_n^{n_1} & \rightarrow & \Omega_C(N) & \rightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

and an exact sequence: $0 \rightarrow B_n^{n_1}[-2] \rightarrow \Omega_B^2(N) \rightarrow \Omega_C^2(N) \rightarrow 0$.

In general there exist exact sequences:

$$0 \rightarrow B_n^{n_{k-1}}[-k] \rightarrow \Omega_B^k(N) \rightarrow \Omega_C^k(N) \rightarrow 0.$$

which induce exact sequences:

$$0 \rightarrow \text{Hom}_{B_n}(\Omega_C^k(N), B_{n,0}) \rightarrow \text{Hom}_{B_n}(\Omega_B^k(N), B_{n,0}) \rightarrow \text{Hom}_{B_n}(B_n^{n_{k-1}}[-k], B_{n,0}) \rightarrow 0.$$

The module $\Omega_C^k(N)$ is annihilated by Z which implies $J_B \Omega_C^k(N) = J_C \Omega_C^k(N)$.

The module $B_{n,0} \cong C_{n,0} \cong \mathbb{k}$

$$\text{Hom}_{B_n}(\Omega_C^k(N), B_{n,0}) \cong \text{Hom}_{B_{n,0}}(\Omega_C^k(N)/J_B \Omega_C^k(N), B_{n,0}) \cong$$

$$\text{Hom}_{C_{n,0}}(\Omega_C^k(N)/J_C \Omega_C^k(N), C_{n,0}) \cong \text{Hom}_{C_n}(\Omega_C^k(N), C_{n,0}) \cong \text{Ext}_{C_n}^k(N, C_{n,0}).$$

We then have an exact sequence: $*) 0 \rightarrow F_{C_n}(N) \xrightarrow{\alpha} F_{B_n}(N) \rightarrow \bigoplus_{k=1}^m S^{n_{k-1}}[k] \rightarrow 0$

Lemma 4. *The map α is a morphism of $C_n^!$ -modules.*

Proof. Let x be an element of $\text{Ext}_{C_n}^k(\mathbb{k}, \mathbb{k})$ and $y \in \text{Ext}_{C_n}^k(N, \mathbb{k})$ we want to prove $\alpha(xy) = x\alpha(y)$.

The element x is an extension: $0 \rightarrow \mathbb{k} \rightarrow E \rightarrow \mathbb{k} \rightarrow 0$ and $y : 0 \rightarrow \mathbb{k} \rightarrow V \rightarrow \Omega_C^{k-1}(N) \rightarrow 0$, the induced map f given below corresponds to y :

$$\begin{array}{ccccccc}
0 \rightarrow & \Omega_C^k(N) & \rightarrow & C_n^{n_{k-1}} & \rightarrow & \Omega_C^{k-1}(N) & \rightarrow 0 \\
& \downarrow f & & \downarrow & & \downarrow 1 & \\
0 \rightarrow & \mathbb{k} & \rightarrow & V & \rightarrow & \Omega_C^{k-1}(N) & \rightarrow 0
\end{array}$$

Consider the following pull back:

$$\begin{array}{ccccccc}
0 \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \Omega_C^k(N) & \rightarrow 0 \\
& \downarrow 1 & & \downarrow & & \downarrow f & \\
0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} & \rightarrow 0
\end{array}$$

The exact sequence: $0 \rightarrow B_n^{n_{k-1}}[-k] \rightarrow \Omega_B^k(N) \xrightarrow{\pi_k} \Omega_C^k(N) \rightarrow 0$ induces a pull back of B_n -modules:

$$\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{k} & \rightarrow & W & \rightarrow & \Omega_B^k(N) \rightarrow 0 \\
& & \downarrow 1 & & \downarrow & & \downarrow \pi_k \\
0 & \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \Omega_C^k(N) \rightarrow 0
\end{array}$$

It was proved above the existence of commutative exact diagrams:

$$\begin{array}{ccccccc}
& & & & 0 & & 0 \\
& & & & \downarrow & & \downarrow \\
& & 0 & \rightarrow & B_n^{n_k-2} & \rightarrow & B_n^{n_k-2} \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Omega_B^k(N) & \rightarrow & B_n^{n_k-2+n_k-1} & \rightarrow & \Omega_B^{k-1}(N) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Omega_B^k(N) & \rightarrow & B_n^{n_k-1} & \rightarrow & \Omega_C^{k-1}(N) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \rightarrow & \Omega_B^k(N) & \rightarrow & B_n^{n_k-1} & \rightarrow & \Omega_C^{k-1}(N) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow 1 \\
0 & \rightarrow & \Omega_C^k(N) & \rightarrow & C_n^{n_k-1} & \rightarrow & \Omega_C^{k-1}(N) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Gluing diagrams we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
& & 0 & \rightarrow & \Omega_B^{k+1}(N) & \rightarrow & B_n^{n_k+n_k-1} \rightarrow B_n^{n_k-2+n_k-1} \rightarrow \Omega_B^{k-1}(N) \rightarrow 0 \\
& & \varphi \downarrow & & \downarrow & & \searrow \downarrow \\
\alpha(xy): & 0 & \rightarrow & \mathbb{k} & \rightarrow & W & \rightarrow B_n^{n_k-1} \rightarrow \Omega_C^{k-1}(N) \rightarrow 0 \\
& & \downarrow 1 & & \downarrow & & \downarrow \\
xy: & 0 & \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow C_n^{n_k-1} \rightarrow \Omega_C^{k-1}(N) \rightarrow 0 \\
& & \downarrow 1 & & \downarrow & & \downarrow \\
xy: & 0 & \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow V \rightarrow \Omega_C^{k-1}(N) \rightarrow 0 \\
& & & & & & \downarrow 1 \\
& & & & & & \Omega_C^{k-1}(N) \rightarrow 0
\end{array}$$

$$\alpha(xy) = \phi \in \text{Ext}_{B_n}^{k+1}(N, K).$$

In the other hand we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \rightarrow & \Omega_B^{k+1}(N) & \rightarrow & B_n^{n_k+n_k-1} & \rightarrow & \Omega_B^k(N) \rightarrow 0 \\
& & \downarrow 1 & & \downarrow & & \downarrow \pi_k \\
0 & \rightarrow & \Omega_B^{k+1}(N) & \rightarrow & B_n^{n_k} & \rightarrow & \Omega_C^k(N) \rightarrow 0 \\
& & \downarrow \pi_{k+1} & & \downarrow & & \downarrow 1 \\
0 & \rightarrow & \Omega_C^{k+1}(N) & \rightarrow & C_n^{n_k} & \rightarrow & \Omega_C^k(N) \rightarrow 0 \\
& & \downarrow \varphi' & & \downarrow & & \downarrow 1 \\
0 & \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \Omega_C^k(N) \rightarrow 0 \\
& & \downarrow 1 & & \downarrow & & \downarrow f \\
0 & \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} \rightarrow 0
\end{array}$$

Gluing diagrams we obtain the following commutative exact diagrams:

$$\begin{array}{ccccccc}
0 & \rightarrow & \Omega_B^{k+1}(N) & \rightarrow & B_n^{n_k+n_k-1} & \rightarrow & \Omega_B^k(N) \rightarrow 0 \\
& & \downarrow \varphi' \pi_{k+1} & & \downarrow & & \downarrow f \pi_k \\
0 & \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} \rightarrow 0 \\
& & & & & & \\
& & 0 & \rightarrow & \Omega_B^{k+1}(N) & \rightarrow & B_n^{n_k+n_k-1} \rightarrow \Omega_B^k(N) \rightarrow 0 \\
& & \downarrow \varphi & & \downarrow & & \downarrow f \pi_k \\
0 & \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} \rightarrow 0
\end{array}$$

The map φ' corresponds with $xf \in \text{Ext}_{C_n}^{k+1}(N, \mathbb{k})$, $\alpha(y) = \pi_k f$, $x\alpha(y) = \varphi$.

Then we have: $\alpha(xy) = \alpha(xf) = \varphi' \pi_{k+1} = \Omega(f \pi_k) = \varphi = x\alpha(y)$. \square

Lemma 5. *There is an isomorphism: $B_n^! F_C(M) = F_B(M)$.*

Proof. Let x be an element of $\text{Ext}_{B_n}^k(M, \mathbb{k})$ and φ the corresponding morphism: $\varphi: \Omega_B^k(M) \rightarrow \mathbb{k}$. As above there exists the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & B_n^{n_k-2}[-k+1] & \rightarrow & B_n^{n_k-2}[-k+1] & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \Omega_B^k(M) & \rightarrow & B_n^{n_k-2+n_{k-1}}[-k+1] & \rightarrow & \Omega_B^{k-1}(M) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \Omega_B \Omega_C^{k-1}(M) & \rightarrow & B_n^{n_k-1}[-k+1] & \rightarrow & \Omega_C^{k-1}(M) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Since the module $\Omega_B^k(M)$ is generated in degree k , there exists an exact sequence of graded modules generated in degree k :

$0 \rightarrow \Omega_B^k(M) \rightarrow JB_n^{n_k-1}[-k+1] \rightarrow J\Omega_C^{k-1}(M) \rightarrow 0$, which in turn induces an exact sequence: $0 \rightarrow J\Omega_B^k(M) \rightarrow J^2 B_n^{n_k-1}[-k+1] \rightarrow J^2 \Omega_C^{k-1}(M) \rightarrow 0$ and there exists a commutative exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & J\Omega_B^k(M) & \rightarrow & J^2 B_n^{n_k-1} & \rightarrow & J^2 \Omega_C^{k-1}(M) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \Omega_B^k(M) & \xrightarrow{j} & JB_n^{n_k-1} & \rightarrow & J\Omega_C^{k-1}(M) & \rightarrow 0 \\
 & \downarrow \pi & & \downarrow \bar{\pi} & & \downarrow & \\
 0 \rightarrow & \Omega_B^k(M)/J\Omega_B^k(M) & \xrightarrow[\bar{q}]{\bar{j}} & JB_n^{n_k-1}/J^2 B_n^{n_k-1} & \rightarrow & J\Omega_C^{k-1}(M)/J^2 \Omega_C^{k-1}(M) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Since $\bar{q}\bar{j}=1$, it follows $\bar{q}\bar{\pi}j=\bar{q}\bar{j}\pi=\pi$.

Being \mathbb{k} semisimple, the map φ factors as follows:

$$\begin{array}{ccc}
 \Omega_B^k(M) & \xrightarrow{\varphi} & \mathbb{k} \\
 \pi \searrow & & \nearrow t \\
 & \Omega_B^k(M)/J\Omega_B^k(M) &
 \end{array}$$

Set $f=t\bar{q}\bar{\pi}$, $f: JB_n^{n_k-1}[-k+1] \rightarrow \mathbb{k}$. Then $fj=t\bar{q}\bar{\pi}j=t\pi=\varphi$.

Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 \rightarrow & \Omega_B \Omega_C^{k-1}(M) & \rightarrow & B_n^{n_k-1} & \rightarrow & \Omega_C^{k-1}(M) & \rightarrow 0 \\
 & \downarrow j & & \downarrow 1 & & \downarrow \rho & \\
 0 \rightarrow & JB_n^{n_k-1}[-k+1] & \rightarrow & B_n^{n_k-1} & \rightarrow & \Omega_C^{k-1}(M)/J\Omega_C^{k-1}(M) & \rightarrow 0 \\
 & \downarrow f & & \downarrow & & \downarrow \cong & \\
 x: 0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \bigoplus_{n_{k-1}} \mathbb{k} & \rightarrow 0
 \end{array}$$

The map corresponding to the last column is: $p = \begin{bmatrix} p_1 \\ \vdots \\ p_{n_{k-1}} \end{bmatrix} : \Omega_C^{k-1}(M) \rightarrow \bigoplus_{n_{k-1}} \mathbb{k}$.

Each $p_i: \Omega_C^{k-1}(M) \rightarrow \mathbb{k}$ corresponds to an element of $\text{Ext}_{C_n}^{k-1}(M, \mathbb{k})$.

$x=(x_1, x_2, \dots, x_{n_{k-1}})$ and each x_i is an extension: $0 \rightarrow \mathbb{k} \rightarrow E_i \rightarrow \mathbb{k} \rightarrow 0$. Taking pull backs:

$$\begin{array}{ccccccc}
x_i p_i: & 0 \rightarrow & \mathbb{k} & \rightarrow & L_i & \rightarrow & \Omega_C^{k-1}(M) \rightarrow 0 \\
& & \downarrow 1 & & \downarrow & & \downarrow p_i \\
x_i: & 0 \rightarrow & \mathbb{k} & \rightarrow & E_i & \rightarrow & \mathbb{k} \rightarrow 0
\end{array}$$

where each $x_i p_i \in B_n^! \text{Ext}_{C_n}^{k-1}(M, \mathbb{k})$ and $x p = \sum x_i p_i \in B_n^! \text{Ext}_{C_n}^{k-1}(M, \mathbb{k})$.

There is also the following induced diagram with exact rows:

$$\begin{array}{ccccccc}
0 \rightarrow & \Omega_B \Omega_C^{k-1}(M) & \rightarrow & B_n^{n_{k-1}}[k+1] & \rightarrow & \Omega_C^{k-1}(M) & \rightarrow 0 \\
& \downarrow h & & \downarrow & & \downarrow 1 & \\
0 \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \Omega_C^{k-1}(M) & \rightarrow 0 \\
& \downarrow 1 & & \downarrow & & \downarrow p & \\
x: 0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \bigoplus_{n_{k-1}} \mathbb{k} & \rightarrow 0
\end{array}$$

Gluing the diagrams we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & \Omega_B \Omega_C^{k-1}(M) & \rightarrow & B_n^{n_{k-1}}[k+1] & \rightarrow & \Omega_C^{k-1}(M) & \rightarrow 0 \\
& \downarrow h & & \downarrow & & \downarrow p & \\
x: 0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \bigoplus_{n_{k-1}} \mathbb{k} & \rightarrow 0
\end{array}$$

It follows $h=fj=\varphi$ up to homotopy.

We have proved $B_n^! \text{Ext}_{C_n}^{k-1}(M, \mathbb{k}) = \text{Ext}_{B_n}^k(M, \mathbb{k})$.

It follows by induction $B_n^! F_C(M) = F_B(M)$. \square

We can prove now the following:

Proposition 4. *Let M be a Koszul non semisimple B_n -module with $ZM=0$, $F_B: K_B \rightarrow K_{B^!}$, $F_C: K_C \rightarrow K_{C^!}$ Koszul dualities. Then there is an isomorphism of $B_n^!$ -modules: $B_n^! \otimes_{C^!} F_C(M) \cong F_B(M)$.*

Proof. We proved in the previous lemma that the map $\mu: B_n^! \otimes_{C^!} F_C(M) \rightarrow F_B(M)$ given by multiplication is surjective and we know that $B_n^! = C_n^! \oplus ZC_n^!$, so there is a splittable sequence of $C_n^!$ -modules: $0 \rightarrow C_n^! \rightarrow B_n^! \rightarrow ZC_n^! \rightarrow 0$ which induces a commutative exact diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & C_n^! \otimes_{C^!} F_C(M) & \rightarrow & B_n^! \otimes_{C^!} F_C(M) & \rightarrow & ZC_n^! \otimes_{C^!} F_C(M) & \rightarrow 0 \\
& \downarrow \cong & & \downarrow \mu & & \downarrow \mu'' & \\
0 \rightarrow & F_C(M) & \rightarrow & F_B(M) & \rightarrow & \bigoplus^m S^{n_{k-1}}[k] & \rightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

By dimensions μ'' is an isomorphism, therefore μ is an isomorphism. \square

It as proved in [MM], [MS] that the duality $\phi: \text{gr}_{B_n^!} \rightarrow \mathcal{LCP}_{B_n}$, with \mathcal{LCP}_{B_n} the category of linear complexes of graded projective B_n -modules, induces a duality of triangulated categories $\bar{\phi}: \underline{\text{gr}}_{B_n^!} \rightarrow D^b(\text{Qgr}_{B_n^{op}})$. In particular given a complex πX° in $D^b(\text{Qgr}_{B_n^{op}})$, there is a totally linear complex (see [MM] for definition) Y° such that πY° is isomorphic to πX° , Moreover, Y° is quasi isomorphic to a linear complex of projective P° and by [MS] $P^\circ = \phi(M)$. Therefore πX° is quasi isomorphic to $\pi \phi(M)$.

It was proved in [MS] that $H^i(\phi(M))=0$ for all $i \neq 0$ if and only if M is Koszul and in this case if $G_{B_n^!}: K_{B_n^!} \rightarrow K_{B_n}$ is Koszul duality, then $H^0(\phi(M)) \cong G_{B_n^!}(M)$.

Since $B_n^!$ is a finite dimensional algebra, it follows by [MZ] that for any finitely generated $B_n^!$ -module M there exists an integer $k \geq 0$ such that $\Omega^k M$ is weakly Koszul.

since Ω is the shift in the triangulated category $\underline{\text{gr}}_{B_n^!}$ and $\bar{\phi}$ is a duality it follows $\bar{\phi}(\Omega^k M) \cong \bar{\phi}(M)[k]$. Being the category \mathcal{T} triangulated, it is invariant under shift and $\pi\phi(M) \in \mathcal{T}$ if and only if $\pi\phi(\Omega^k M) \in \mathcal{T}$.

We may assume M is weakly Koszul and $M = \sum_{i \geq 0} M_i$, $M_0 \neq 0$. By [MZ] there exists an exact sequence: $0 \rightarrow K_M \rightarrow M \rightarrow L \rightarrow 0$ with $K_M < M_0 >$ generated by the degree zero part of M , K_M is Koszul and $J^j K_M = J^j M \cap K_M$ for all $j > 0$.

Being ϕ an exact functor there is an exact sequence: $0 \rightarrow \phi(L) \rightarrow \phi(M) \rightarrow \phi(K_M) \rightarrow 0$ of complexes of B_n -modules, which induces a long exact sequence:

$$\dots \rightarrow H^1(\phi(L)) \rightarrow H^1(\phi(M)) \rightarrow H^1(\phi(K_M)) \rightarrow H^0(\phi(L)) \rightarrow H^0(\phi(M)) \rightarrow H^0(\phi(K_M)) \rightarrow 0$$

where $H^0(\phi(M)) \cong H^0(\phi(K_M))$ and $H^i(\phi(L)) \cong H^i(\phi(M))$ for all $i \neq 0$. Being K_M Koszul $H^0(\phi(K_M)) \cong G_{B_n^!}(K_M)$ and $G_{B_n^!}(K_M)$ is of \mathbb{Z} -torsion.

According to [MZ] there is a filtration: $M = U_p \supset U_{p-1} \supset \dots \supset U_1 \supset U_0 = K_M$ such that U_i/U_{i-1} is Koszul and $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.

The module L is weakly Koszul and it has a filtration: $L = U_p/U_0 \supset U_{p-1}/U_0 \supset \dots \supset U_1/U_0$ with factors Koszul, it follows by induction each $G_{B_n^!}(U_i/U_{i-1}) = V_i$ is a Koszul B_n -module of \mathbb{Z} -torsion.

Each V_i has a filtration: $V_i \supset ZV_i \supset Z^2V_i \supset \dots \supset Z^{k_i}V_i \supset 0$, $Z^{k_i}V_i \neq 0$, $Z^{k_i+1}V_i = 0$. After a truncation $V_{i \geq n_i}$ we may assume all $Z^j V_i$ Koszul. But $V_{i \geq n_i} = J^{n_i} V_i \cong G_{B_n^!}(\Omega^{n_i}(U_i/U_{i-1}))$. Taking $n = \max\{n_i\}$ we change M for $\Omega^n(M)$, which is weakly Koszul with filtration: $\Omega^n M = \Omega^n U_p \supset \Omega^n U_{p-1} \supset \dots \supset \Omega^n U_1 \supset \Omega^n U_0$.

We may assume all $Z^j V_i$ are Koszul. There exist exact sequences:

$$*) 0 \rightarrow F_{B_n}(Z^{k_i} V_i)[-k_i] \rightarrow F_{B_n}(Z^{k_i-1} V_i/Z^{k_i} V_i)[-k_i+1] \rightarrow \dots \rightarrow$$

$$F_{B_n}(V_i/ZV_i) \rightarrow U_i/U_{i-1} \rightarrow 0$$

where each $F_{B_n}(Z^j V_i/Z^{j+1} V_i) \cong B_n^! \otimes_{C_n^!} X_{ij}$ is an induced module of a Koszul

$C_n^!$ -module X_{ij} .

Lemma 6. *Let R be a \mathbb{Z} graded \mathbb{k} -algebra, with \mathbb{k} a field, M a graded left R -module and N a graded right R -module. Then $M \otimes_R N$ is a graded \mathbb{k} -module such that $M \otimes_R N[j] \cong (M \otimes_R N)[j]$ as graded \mathbb{k} -modules.*

Proof. Recall the definition of the graded tensor product [Mac]:

Let $\psi: M \otimes_{\mathbb{k}} R \otimes_{\mathbb{k}} N \rightarrow M \otimes_{\mathbb{k}} N$ be the map: $\psi(m \otimes r \otimes n) = mr \otimes n - m \otimes rn$. Then $\text{Cok} \psi = M \otimes_R N$.

The \mathbb{k} -module $M \otimes_R N$ has grading: $(M \otimes_R N)_k = \sum_{i+j=k} M_i \otimes_{\mathbb{k}} N_j$. It follows $M \otimes_R N[j] \cong (M \otimes_R N)[j]$.

and there is an isomorphism of exact sequences:

$$\begin{array}{ccccccc} M \otimes_{\mathbb{k}} R \otimes_{\mathbb{k}} (N[j]) & \rightarrow & M \otimes_{\mathbb{k}} (N[j]) & \rightarrow & M \otimes_{\mathbb{k}} (N[j]) & \rightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ (M \otimes_{\mathbb{k}} R \otimes_{\mathbb{k}} N)[j] & \rightarrow & (M \otimes_{\mathbb{k}} N)[j] & \rightarrow & (M \otimes_R N)[j] & \rightarrow & 0 \end{array}$$

□

Lemma 7. *Let $B_n^!$ and $C_n^!$ be the algebras given above, for any finitely generated graded $C_n^!$ -module M there is an isomorphism: $\Omega_{B_n^!}(B_n^! \otimes_{C_n^!} M) \cong B_n^! \otimes_{C_n^!} \Omega_{C_n^!}(M)$.*

Proof. Let $0 \rightarrow \Omega_{C_n^!}(M) \rightarrow F \rightarrow M \rightarrow 0$ be an exact sequence with F free of rank r , the graded projective cover of M . Then $\Omega_{C_n^!}(M) \subset J_{C_n^!}F$.

We proved $B_n^! = C_n^! \oplus ZC_n^!$, therefore $J_{B_n^!} = J_{C_n^!} + ZC_n^!$. It follows: $B_n^! \otimes_{C_n^!} J_{C_n^!} = C_n^! \otimes_{C_n^!} J_{C_n^!} + ZC_n^! \otimes_{C_n^!} J_{C_n^!} = J_{C_n^!} + ZC_n^! \otimes_{C_n^!} J_{C_n^!} = J_{C_n^!} + Z \otimes_{C_n^!} J_{C_n^!} \subset J_{B_n^!}$.

Therefore: $B_n^! \otimes_{C_n^!} \Omega_{C_n^!}(M) \subset B_n^! \otimes_{C_n^!} J_{C_n^!} \cong \bigoplus_r B_n^! \otimes_{C_n^!} J_{C_n^!} \subset \bigoplus_r J_{B_n^!} \cong J_{B_n^!}(B_n^! \otimes_{C_n^!} F)$.

It follows: $0 \rightarrow B_n^! \otimes_{C_n^!} \Omega_{C_n^!}(M) \rightarrow B_n^! \otimes_{C_n^!} F \rightarrow B_n^! \otimes_{C_n^!} M \rightarrow 0$ is exact and $B_n^! \otimes_{C_n^!} F$ is the graded projective cover of $B_n^! \otimes_{C_n^!} M$. Then $\Omega_{B_n^!}(B_n^! \otimes_{C_n^!} M) \cong B_n^! \otimes_{C_n^!} \Omega_{C_n^!}(M)$. \square

Lemma 8. *Let $B_n^!$ and $C_n^!$ be the algebras given above and let M be a Koszul $C_n^!$ -module. Then $B_n^! \otimes_{C_n^!} M$ is Koszul and $G_{B_n^!}(B_n^! \otimes_{C_n^!} M) \cong G_{C_n^!}(M)$.*

Proof. Let $\dots \rightarrow F_n[-n] \rightarrow F_{n-1}[-n+1] \rightarrow \dots \rightarrow F_1[-1] \rightarrow F_0 \rightarrow M \rightarrow 0$ be a graded projective resolution of M , with each F_i free of rank r_i . Tensoring with $B_n^! \otimes_{C_n^!}$ we obtain a graded projective resolution of $B_n^! \otimes_{C_n^!} M$: $\rightarrow (B_n^! \otimes_{C_n^!} F_n)[-n] \rightarrow (B_n^! \otimes_{C_n^!} F_{n-1})[-n+1] \rightarrow \dots \rightarrow (B_n^! \otimes_{C_n^!} F_1)[-1] \rightarrow B_n^! \otimes_{C_n^!} F_0 \rightarrow B_n^! \otimes_{C_n^!} M \rightarrow 0$ with each $B_n^! \otimes_{C_n^!} F_i$ free $B_n^!$ -modules of rank r_i .

Moreover, $\text{Ext}_{B_n^!}^n(B_n^! \otimes_{C_n^!} M, \mathbb{k}) \cong \text{Hom}_{B_n^!}(\Omega^n(B_n^! \otimes_{C_n^!} M), \mathbb{k}) \cong$

$\text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} \Omega^n M, \mathbb{k}) \cong \text{Hom}_{C_n^!}(\Omega^n M, \mathbb{k}) \cong \text{Ext}_{C_n^!}^n(M, \mathbb{k})$.

Therefore: $G_{B_n^!}(B_n^! \otimes_{C_n^!} M) \cong G_{C_n^!}(M)$. \square

Remark 2. *To be $G_{B_n^!}(B_n^! \otimes_{C_n^!} M)$ a C_n -module means: $ZG_{B_n^!}(B_n^! \otimes_{C_n^!} M) = 0$.*

We know $B_n \cong \bigoplus_{m \geq 0} \text{Ext}_{B_n^!}^m(\mathbb{k}, \mathbb{k})$, $C_n \cong \bigoplus_{m \geq 0} \text{Ext}_{C_n^!}^m(\mathbb{k}, \mathbb{k})$, and $B_n/ZB_n \cong C_n$. Since $C_n^!$ is a sub algebra of $B_n^!$, given an extension $x: 0 \rightarrow \mathbb{k} \rightarrow E_1 \rightarrow E_2 \rightarrow \dots \rightarrow E_n \rightarrow \mathbb{k} \rightarrow 0$ of $B_n^!$, we obtain by restriction of scalars an extension $\text{res}x: 0 \rightarrow \text{res}\mathbb{k} \rightarrow \text{res}E_1 \rightarrow \text{res}E_2 \rightarrow \dots \rightarrow \text{res}E_n \rightarrow \text{res}\mathbb{k} \rightarrow 0$ of $C_n^!$ -modules, where $\text{res}M$ is the module M with multiplication of scalars restricted to $C_n^!$. It is clear $\text{res}(xy) = \text{res}(x)\text{res}(y)$ and restriction gives an homomorphism of graded \mathbb{k} -algebras: $\text{res}: \bigoplus_{m \geq 0} \text{Ext}_{B_n^!}^m(\mathbb{k}, \mathbb{k}) \rightarrow \bigoplus_{m \geq 0} \text{Ext}_{C_n^!}^m(\mathbb{k}, \mathbb{k})$.

Lemma 9. *There is an homomorphism: $\rho: \text{Ext}_{B_n^!}^1(\mathbb{k}, \mathbb{k}) \rightarrow \text{Ext}_{B_n^!}^1((B_n^! \otimes_{C_n^!} \mathbb{k}, \mathbb{k}))$, given by the Yoneda product $\rho(x) = x\mu$ (pull back) of the exact sequence x with the multiplication map $\mu: B_n^! \otimes_{C_n^!} \mathbb{k} \rightarrow \mathbb{k}$, such that the composition of the map, $\psi_1: \text{Ext}_{B_n^!}^1((B_n^! \otimes_{C_n^!} \mathbb{k}, \mathbb{k})) \rightarrow \text{Ext}_{C_n^!}^1(\mathbb{k}, \mathbb{k})$ in the previous lemma with ρ , is the restriction: $\psi\rho = \text{res}$.*

Proof. Let x be the extension: $x: 0 \rightarrow \mathbb{k} \rightarrow E \rightarrow \mathbb{k} \rightarrow 0$. Since $B_n^!$ is a free $C_n^!$ -module, there is a commutative exact diagram

$$\begin{array}{ccccccc} 0 \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow & B_n^! \otimes_{C_n^!} E & \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow 0 \\ & \downarrow \mu & & \downarrow \mu & & \downarrow \mu & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} & \rightarrow 0 \end{array}$$

with μ multiplication.

This diagram splits in two diagrams:

$$\begin{array}{ccccccc} 0 \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow & B_n^! \otimes_{C_n^!} E & \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow 0 \\ & \downarrow \mu & & \downarrow & & \downarrow 1 & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & W & \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow 0 \\ & \downarrow 1 & & \downarrow & & \downarrow \mu & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} & \rightarrow 0 \end{array}$$

Then $\rho(x) = x\mu = \mu(B_n^! \otimes x)$.

For any finitely generated $C_n^!$ -module M there is an isomorphism α obtained as the composition of the natural isomorphisms:

$$\mathrm{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} M, \mathbb{k}) \cong \mathrm{Hom}_{C_n^!}(M, \mathrm{Hom}_{B_n^!}(B_n^!, \mathbb{k})) \cong \mathrm{Hom}_{C_n^!}(M, \mathbb{k}).$$

If $j: M \rightarrow B_n^! \otimes_{C_n^!} M$ be the map $j(m) = 1 \otimes m$ and $f: B_n^! \otimes_{C_n^!} M \rightarrow \mathbb{k}$ is any map, then $\alpha(f) = fj$.

Then $\psi\rho(x) = \psi(x\mu)$ is the top sequence in the commutative exact diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \mathbb{k} & \rightarrow 0 \\ & \downarrow 1 & & \downarrow & & \downarrow j & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & W & \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow 0 \\ & \downarrow 1 & & \downarrow & & \downarrow \mu & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & E & \rightarrow & \mathbb{k} & \rightarrow 0 \end{array}$$

Since $\mu j = 1$, gluing both diagrams we obtain $\psi\rho(x) = \mathrm{res}x$. \square

Lemma 10. *Under the conditions of the previous lemma the map ρ is surjective.*

Proof. Since $B_n^! = C_n^! \oplus ZC_n^!$, $B_n^! \otimes_{C_n^!} \mathbb{k}$ is a graded vector space of dimension two with one copy of \mathbb{k} in degree zero and one copy of \mathbb{k} in degree one. Hence the multiplication map $\mu: B_n^! \otimes_{C_n^!} \mathbb{k} \rightarrow \mathbb{k}$ is an epimorphism with kernel $u: \mathbb{k}[-1] \rightarrow B_n^! \otimes_{C_n^!} \mathbb{k}$.

Let $y: 0 \rightarrow \mathbb{k}[-1] \rightarrow E \rightarrow B_n^! \otimes_{C_n^!} \mathbb{k} \rightarrow 0$ be an element of $\mathrm{Ext}_{B_n^!}^1(\mathbb{k}, B_n^! \otimes_{C_n^!} \mathbb{k})$ and take the pullback:

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbb{k}[-1] & \rightarrow & N & \rightarrow & \mathbb{k}[-1] & \rightarrow 0 \\ & \downarrow 1 & & \downarrow & & \downarrow u & \\ 0 \rightarrow & \mathbb{k}[-1] & \rightarrow & E & \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow 0 \end{array}$$

But the top exact sequence splits because the ends are generated in the same degree and the algebra is Koszul or equivalently there is a lifting $v: \mathbb{k}[-1] \rightarrow E$ of u and we get a commutative exact diagram:

$$\begin{array}{ccccccc}
& 0 & \rightarrow & \mathbb{k}[-1] & \rightarrow & \mathbb{k}[-1] & \rightarrow 0 \\
& \downarrow & & \downarrow v & & \downarrow u & \\
0 \rightarrow & \mathbb{k}[-1] & \rightarrow & E & \rightarrow & B_n^! \otimes_{C_n^!} \mathbb{k} & \rightarrow 0 \\
& \downarrow 1 & & \downarrow & & \downarrow \mu & \\
0 \rightarrow & \mathbb{k}[-1] & \rightarrow & L & \rightarrow & \mathbb{k} & \rightarrow 0
\end{array}$$

Proving ρ is surjective. \square

Corollary 5. *The map $\text{res}: \bigoplus_{m \geq 0} \text{Ext}_{B_n^!}^m(\mathbb{k}, \mathbb{k}) \rightarrow \bigoplus_{m \geq 0} \text{Ext}_{C_n^!}^m(\mathbb{k}, \mathbb{k})$ is a surjective homomorphism of algebras and the kernel of res is the ideal $ZB_n = B_n Z$.*

Proof. Since both $B_n^!$ and $C_n^!$ are Koszul algebras they are graded algebras generated in degree one, and it follows from the lemma that for any $m > 0$ the map $\text{res}: \text{Ext}_{B_n^!}^m(\mathbb{k}, \mathbb{k}) \rightarrow \text{Ext}_{C_n^!}^m(\mathbb{k}, \mathbb{k})$ is surjective.

Observe that for any homomorphism $f: B_n \rightarrow C_n$ Z is in the kernel.

We have in B_n the equality $X_1 \delta_1 - \delta_1 X_1 = Z^2$. Since C_n is commutative, $f(X_1 \delta_1 - \delta_1 X_1) = f(X_1) f(\delta_1) - f(\delta_1) f(X_1) = f(Z)^2 = 0$,

But since C_n is an integral domain, it follows $f(Z) = 0$.

In particular $ZB_n \subseteq \text{Ker}(\text{res})$ and there is a factorization:

$$\begin{array}{ccc}
B_n & \rightarrow & C_n \\
\searrow & & \nearrow \alpha \\
& B_n/ZB_n &
\end{array}$$

and since $B_n/ZB_n \cong C_n$ it follows by dimension, that α is an isomorphism. \square

Lemma 11. *With the same notation as in the previous lemma, let M be a Koszul $C_n^!$ -module and $\psi: G_{B_n^!}(B_n^! \otimes_{C_n^!} M) \rightarrow G_{C_n^!}(M)$, the isomorphism in the previous lemma.*

Then given $y \in \text{Ext}_{B_n^!}^m(B_n^! \otimes_{C_n^!} M, \mathbb{k})$ and $c \in \text{Ext}_{B_n^!}^1(\mathbb{k}, \mathbb{k})$, we have $\psi(cy) = \text{res}(c)\psi(y)$.

Proof. The map $f: B_n^! \otimes_{C_n^!} \Omega^m(M) \rightarrow \mathbb{k}$ corresponding to the extension y is the map in the commutative exact diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & B_n^! \otimes_{C_n^!} \Omega^m(M) & \rightarrow & B_n^! \otimes_{C_n^!} C_n^{!k_{m-1}} & \rightarrow & \dots \rightarrow & B_n^! \otimes_{C_n^!} C_n^{!k_0} \rightarrow B_n^! \otimes_{C_n^!} M \rightarrow 0 \\
& \downarrow f & & \downarrow & & & \downarrow & \downarrow 1 \\
0 \rightarrow & \mathbb{k} & \rightarrow & E_1 & \rightarrow & \dots \rightarrow & E_m \rightarrow B_n^! \otimes_{C_n^!} M \rightarrow 0
\end{array}$$

where y is the bottom row.

If $j: \Omega^m(M) \rightarrow B_n^! \otimes_{C_n^!} \Omega^m(M)$ is the map $j(m) = 1 \otimes m$, then $\psi(y)$ is the extension corresponding to the map fj .

Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \rightarrow & \Omega^{m+1}(M) & \rightarrow & C_n^{!k_m} & \rightarrow & \Omega^m(M) & \rightarrow 0 \\
& \downarrow j & & \downarrow j & & \downarrow j & \\
0 \rightarrow & B_n^! \otimes_{C_n^!} B\Omega^{m+1}(M) & \rightarrow & B_n^! \otimes_{C_n^!} C_n^{!k_m} & \rightarrow & B_n^! \otimes_{C_n^!} \Omega^m(M) & \rightarrow 0 \\
& \downarrow \Omega f & & \downarrow & & \downarrow f & \\
0 \rightarrow & JB_n^! & \rightarrow & B_n^! & \rightarrow & \mathbb{k} & \rightarrow 0 \\
& \downarrow g & & \downarrow & & \downarrow 1 & \\
0 \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \mathbb{k} & \rightarrow 0
\end{array}$$

where c is the bottom sequence.

Since $B_n^! = C_n^! \oplus C_n^! Z$ as $C_n^!$ -module, the map g restricted to $C_n^!$ represents the extension $\text{res}(c)$.

Taking the pullback we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow & \Omega^{m+1}(M) & \rightarrow & C_n^{!k_m} & \rightarrow & \Omega^m(M) & \rightarrow 0 \\ & \text{g}\Omega(f)j \downarrow & & \downarrow & & 1 \downarrow & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & W & \rightarrow & \Omega^m(M) & \rightarrow 0 \\ & 1 \downarrow & & \downarrow & & f j \downarrow & \\ 0 \rightarrow & \mathbb{k} & \rightarrow & L & \rightarrow & \mathbb{k} & \rightarrow 0 \end{array}$$

and $\Omega(fj) = \Omega(f)j$.

It follows $\psi(cy) = \text{res}(c)\psi(y)$. \square

As a corollary we obtain the following:

Proposition 5. *Let M be a Koszul $C_n^!$ -module and $G_{B_n^!} = \bigoplus_{m \geq 0} \text{Ext}_{C_n^!}^m(-, \mathbb{k})$ Koszul duality. Then $Z(G_{B_n^!}(B_n^! \otimes_{C_n^!} M)) = 0$.*

Proof. Denote by z the extension corresponding to Z under the isomorphism $B_n \cong \bigoplus_{m \geq 0} \text{Ext}_{B_n^!}^m(\mathbb{k}, \mathbb{k})$. By the previous lemma, for any extension $y \in \text{Ext}_{B_n^!}^m(B_n^! \otimes_{C_n^!} M, \mathbb{k})$, $\psi(zy) = \text{res}(z)\psi(y)$ and by lemma 10, $\text{res}(z) = 0$. Since ψ is an isomorphism, it follows $zy = 0$, hence $Z(G_{B_n^!}(B_n^! \otimes_{C_n^!} M)) = 0$. \square

Proposition 6. *Let $B_n^!$ and $C_n^!$ be the algebras given above. Then for any induced module $B_n^! \otimes_{C_n^!} M$ when we apply the duality $\bar{\phi}$ to $B_n^! \otimes_{C_n^!} M$ we obtain an element of \mathcal{T} .*

Proof. There exists some integer $n \geq 0$ such that $\Omega^n M$ and $\Omega^n(B_n^! \otimes_{C_n^!} M)$ are weakly Koszul. Since $\bar{\phi}(\Omega^n(B_n^! \otimes_{C_n^!} M)) \cong \bar{\phi}(B_n^! \otimes_{C_n^!} M)[n]$. The object $\bar{\phi}(\Omega^n(B_n^! \otimes_{C_n^!} M))$ is in \mathcal{T} if and only if $\bar{\phi}(B_n^! \otimes_{C_n^!} M)$ is in \mathcal{T} . We may assume M and $B_n^! \otimes_{C_n^!} M$ are weakly Koszul.

The module M has a filtration: $M = U_p \supset U_{p-1} \supset \dots \supset U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, hence; $B_n^! \otimes_{C_n^!} M$ has a filtration: $B_n^! \otimes_{C_n^!} M = B_n^! \otimes_{C_n^!} U_p \supset B_n^! \otimes_{C_n^!} U_{p-1} \supset \dots \supset B_n^! \otimes_{C_n^!} U_1 \supset B_n^! \otimes_{C_n^!} U_0$ such that $B_n^! \otimes_{C_n^!} U_i/B_n^! \otimes_{C_n^!} U_{i-1} \cong B_n^! \otimes_{C_n^!} U_i/U_{i-1}$ is Koszul.

The exact sequence: $0 \rightarrow B_n^! \otimes_{C_n^!} U_0 \rightarrow B_n^! \otimes_{C_n^!} U_1 \rightarrow B_n^! \otimes_{C_n^!} U_1/U_0 \rightarrow 0$ induces an exact sequence of complexes:

$$\begin{aligned} 0 \rightarrow & \phi(B_n^! \otimes_{C_n^!} U_1/U_0) \rightarrow \phi(B_n^! \otimes_{C_n^!} U_1) \rightarrow \phi(B_n^! \otimes_{C_n^!} U_0) \rightarrow 0 \text{ which in turn induces a} \\ & \text{long exact sequence:} \\ \dots \rightarrow & H^1(\phi(B_n^! \otimes_{C_n^!} U_1/U_0)) \rightarrow H^1(\phi(B_n^! \otimes_{C_n^!} U_1)) \rightarrow H^1(\phi(B_n^! \otimes_{C_n^!} U_0)) \rightarrow H^0(\phi(B_n^! \otimes_{C_n^!} U_1/U_0)) \\ \rightarrow & H^0(\phi(B_n^! \otimes_{C_n^!} U_1)) \rightarrow H^0(\phi(B_n^! \otimes_{C_n^!} U_0)) \rightarrow 0 \end{aligned}$$

where $H^i(\phi(B_n^! \otimes_{C_n^!} U_0)) = 0$ for $i \neq 0$ and $H^0(\phi(B_n^! \otimes_{C_n^!} U_0)) = G_{B_n^!}(B_n^! \otimes_{C_n^!} U_0) \cong G_{C_n^!}(U_0)$ of Z -torsion, $H^0(\phi(B_n^! \otimes_{C_n^!} U_1)) \cong H^0(\phi(B_n^! \otimes_{C_n^!} U_0))$ and $H^i(\phi(B_n^! \otimes_{C_n^!} U_1/U_0)) \cong$

$H^i(\phi(B_n^! \otimes_{C_n^!} U_1))$ for $i \neq 0$. It follows $H^i(\phi(B_n^! \otimes_{C_n^!} U_1))$ is of \mathbb{Z} -torsion for all i . By induction $H^i(\phi(B_n^! \otimes_{C_n^!} M))$ is of \mathbb{Z} -torsion for all i .

We have proved $\phi(B_n^! \otimes_{C_n^!} M) \in \mathcal{T}$. \square

Lemma 12. *Let M be a $B_n^!$ -module and assume there is an integer $n \geq 0$ such that $\Omega^n M = N$ has the following properties:*

The module N is weakly Koszul, it has a filtration: $N = U_p \supset U_{p-1} \supset \dots \supset U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, and for all $k \geq 0$, $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.

The Koszul modules $G_{B_n^!}(U_i/U_{i-1}) = V_i$ are of \mathbb{Z} -torsion.

Then $\phi(M)$ is in \mathcal{T} .

Proof. As above, $\phi(M)$ is in \mathcal{T} if and only if $\phi(N)$ is in \mathcal{T} .

The exact sequence: $0 \rightarrow U_0 \rightarrow U_1 \rightarrow U_1/U_0 \rightarrow 0$ induces an exact sequence: $0 \rightarrow \phi(U_1/U_0) \rightarrow \phi(U_1) \rightarrow \phi(U_0) \rightarrow 0$ such that $H^0(\phi(U_1)) \cong H^0(\phi(U_0)) \cong G_{B_n^!}(U_0)$ is of \mathbb{Z} -torsion and $H^i(\phi(U_1/U_0)) \cong H^i(\phi(U_1))$ is of \mathbb{Z} -torsion for all $i \neq 0$. By induction, $H^i(\phi(N))$ is of \mathbb{Z} -torsion for all i , hence $\phi(N)$ is in \mathcal{T} . \square

Theorem 2. *Let \mathcal{T}' be the subcategory of $\text{gr}_{B_n^!}$ corresponding to \mathcal{T} under the duality: $\bar{\phi}: \text{gr}_{B_n^!} \rightarrow D^b(Q\text{gr}_{B_n})$. This is: $\bar{\phi}(\mathcal{T}') = \mathcal{T}$. Then \mathcal{T}' is the smallest triangulated subcategory of $\text{gr}_{B_n^!}$ containing the induced modules and closed under the Nakayama automorphism.*

Proof. Let \mathcal{B} be a triangulated subcategory of $\text{gr}_{B_n^!}$ containing the induced modules and closed under the Nakayama automorphism. Let $M \in \mathcal{T}'$ and $\Omega^n M = N$ weakly Koszul with a filtration $N = U_p \supset U_{p-1} \supset \dots \supset U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, and for all $k \geq 0$, $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.

Since \mathcal{T}' is closed under the shift the module N is also in \mathcal{T}' . We prove by induction on the length of the filtration that for each i the modules $U_i, U_i/U_{i-1}$ are in \mathcal{T}' .

We have an exact sequence of complexes: $0 \rightarrow \phi(N/U_0) \rightarrow \phi(N) \rightarrow \phi(U_0) \rightarrow 0$

By the long homology sequence there is an exact sequence:

$$\dots \rightarrow H_{i+1}(\phi(U_0)) \rightarrow H_i(\phi(N/U_0)) \rightarrow H_i(\phi(N)) \rightarrow H_i(\phi(U_0)) \rightarrow H_{i-1}(\phi(N/U_0)) \rightarrow \dots \rightarrow H_0(\phi(N/U_0)) \rightarrow H_0(\phi(N)) \rightarrow H_0(\phi(U_0)) \rightarrow 0$$

By [MZ], $H_0(\phi(N)) = H_0(\phi(U_0))$, $H_0(\phi(N/U_0)) = 0$ and $H_i(\phi(U_0)) = 0$ for all $i \neq 0$. Then $H_i(\phi(N/U_0)) = H_i(\phi(N))$ for all $i \neq 0$ and $H_0(\phi(U_0))$ is of \mathbb{Z} -torsion and $H_i(\phi(N/U_0))$ is of \mathbb{Z} -torsion for all i .

It follows by induction, $U_i, U_i/U_{i-1}$ are in \mathcal{T}' for all i .

The Koszul modules $G_{B_n^!}(U_i/U_{i-1}) = V_i$ are of \mathbb{Z} -torsion and each $Z^j V_i$ is Koszul.

There exists an exact sequence:

$$0 \rightarrow F_{B_n}(Z^{k_i} V_i)[-k_i] \rightarrow F_{B_n}(Z^{k_i-1} V_i/Z^{k_i} V_i)[-k_i+1] \dots$$

$$\rightarrow F_{B_n}(V_i/ZV_i) \rightarrow U_i/U_{i-1} \rightarrow 0 \text{ where each } F_{B_n}(Z^j V_i/Z^{j+1} V_i) \cong B_n^! \otimes_{C_n^!} X_{ij} \text{ is an}$$

induced module of a Koszul $C_n^!$ -module X_{ij} .

Then each $F_{B_n}(Z^j V_i/Z^{j+1} V_i) \cong B_n^! \otimes_{C_n^!} X_{ij}$ is in \mathcal{B} .

Moreover, the exact sequences: $0 \rightarrow B_n^! \otimes_{C_n^!} X_{ik_i} \rightarrow B_n^! \otimes_{C_n^!} X_{ik_i-1} \rightarrow K_{k_i-2} \rightarrow 0$ gives rise to triangles: $B_n^! \otimes_{C_n^!} X_{ik_i} \rightarrow B_n^! \otimes_{C_n^!} X_{ik_i-1} \rightarrow K_{k_i-2} \rightarrow \Omega^{-1}(B_n^! \otimes_{C_n^!} X_{ik_i})$. Therefore $K_{k_i-2} \in \mathcal{B}$. It follows by induction, $U_i/U_{i-1} \in \mathcal{B}$.

The filtration $N=U_p \supset U_{p-1} \supset \dots \supset U_1 \supset U_0$ induces triangles:

$U_0 \rightarrow U_1 \rightarrow U_1/U_0 \rightarrow \Omega^{-1}(U_0)$ with $U_0, U_1/U_0 \in \mathcal{B}$. It follows $U_1 \in \mathcal{B}$.

By induction, $N \in \mathcal{B}$.

We have proved $\mathcal{T}' \subset \mathcal{B}$. \square

Theorem 3. *Let \mathcal{T}' be the subcategory of $\underline{\text{gr}}_{B_n^!}$ corresponding to \mathcal{T} under the duality: $\overline{\phi}: \underline{\text{gr}}_{B_n^!} \rightarrow D^b(Q\text{gr}_{B_n})$. This is: $\overline{\phi}(\mathcal{T}') = \mathcal{T}$. Then \mathcal{T}' has Auslander Reiten triangles and they are of type $\mathbb{Z}A_\infty$.*

Proof. Let M be an indecomposable non projective module in \mathcal{T}' . Then we have almost split sequences: $0 \rightarrow \sigma\Omega^2 M \rightarrow E \rightarrow M \rightarrow 0$ and $0 \rightarrow M \rightarrow F \rightarrow \sigma^{-1}\Omega^{-2}M \rightarrow 0$, since the category \mathcal{T} is closed under the Nakayama automorphism, \mathcal{T}' is also closed under the Nakayama automorphism and $\sigma\Omega^2 M, \sigma^{-1}\Omega^{-2}M$ are objects in \mathcal{T}' . From the exact sequences of complexes: $0 \rightarrow \phi(M) \rightarrow \phi(E) \rightarrow \phi(\sigma\Omega^2 M) \rightarrow 0$ and $0 \rightarrow \phi(\sigma^{-1}\Omega^{-2}M) \rightarrow \phi(F) \rightarrow \phi(M) \rightarrow 0$ and the long homology sequence we get that both $\phi(E)$ and $\phi(F)$ are in \mathcal{T} . Therefore: E and F are in \mathcal{T}' . We have proved \mathcal{T}' has almost split sequences and they are almost split sequences in $\text{gr}_{B_n^!}$. We proved in [MZ] that the Auslander Reiten components of $\underline{\text{gr}}_{B_n^!}$ are of type $\mathbb{Z}A_\infty$. It follows $\sigma\Omega^2 M \rightarrow E \rightarrow M \rightarrow \sigma\Omega^2 M[-1], M \rightarrow F \rightarrow \sigma^{-1}\Omega^{-2}M \rightarrow M[-1]$ are Auslander Reiten triangles and the Auslander Reiten components are of type $\mathbb{Z}A_\infty$. \square

We will characterize now the full subcategory \mathcal{F}' of $\underline{\text{gr}}_{B_n^!}$ such that $\overline{\phi}(\mathcal{F}') = \mathcal{F}$.

Theorem 4. *The subcategory \mathcal{F}' of $\underline{\text{gr}}_{B_n^!}$ such that $\overline{\phi}(\mathcal{F}') = \mathcal{F}$ consists of the graded $B_n^!$ -modules M such that the restriction of M to $C_n^!$ is injective.*

Proof. Let $M \in \mathcal{F}'$. There is an isomorphism:

$\text{Hom}_{D^b(Q\text{gr}_{B_n^{\text{op}}})}(\mathcal{T}, \pi\phi(M)) \cong \underline{\text{Hom}}_{\text{gr}_{B_n^!}}(M, \mathcal{T}') = 0$, which implies $\underline{\text{Hom}}_{B_n^!}(M, \mathcal{T}') = 0$

In particular for any induced module $B_n^! \otimes_{C_n^!} \Omega^2 L$ we have:

$$\underline{\text{Hom}}_{B_n^!}(M, B_n^! \otimes_{C_n^!} \Omega^2 L) = 0 = \underline{\text{Hom}}_{B_n^!}(\Omega^{-2}M, B_n^! \otimes_{C_n^!} L).$$

By Auslander-Reiten formula:

$$D(\underline{\text{Hom}}_{B_n^!}(\Omega^{-2}M, B_n^! \otimes_{C_n^!} L)) = \text{Ext}_{B_n^!}^1(B_n^! \otimes_{C_n^!} L, M) = 0 \text{ for all } L \in \text{gr}_{C_n^!}.$$

Consider the exact sequences: $0 \rightarrow \Omega_{C_n^!}(L) \rightarrow F \rightarrow L \rightarrow 0$, with F the projective cover of L . It induces an exact sequence:

$$0 \rightarrow B_n^! \otimes_{C_n^!} \Omega_{C_n^!}(L) \rightarrow B_n^! \otimes_{C_n^!} F \rightarrow B_n^! \otimes_{C_n^!} L \rightarrow 0$$

By the long homology sequence, there is an exact sequence:

$$0 \rightarrow \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} L, M) \rightarrow \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} F, M) \rightarrow \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} \Omega_{C_n^!}(L), M) \rightarrow$$

$\text{Ext}_{B_n^!}^1(B_n^! \otimes_{C_n^!} L, M) \rightarrow 0$ which by the adjunction isomorphism are isomorphic to

the exact sequences:

$$0 \rightarrow \text{Hom}_{C_n^!}(L, M) \rightarrow \text{Hom}_{C_n^!}(F, M) \rightarrow \text{Hom}_{C_n^!}(\Omega_{C_n^!}(L), M) \rightarrow \text{Ext}_{C_n^!}^1(L, M) \rightarrow 0.$$

It follows, $\text{Ext}_{C_n^!}^1(L, M) \cong \text{Ext}_{B_n^!}^1(B_n^! \otimes_{C_n^!} L, M)$ and by dimension shift

$$\text{Ext}_{C_n^!}^k(L, M) \cong \text{Ext}_{B_n^!}^k(B_n^! \otimes_{C_n^!} L, M) = 0 \text{ for all } k \geq 1.$$

We have proved the restriction of M to $C_n^!$ is injective.

Let's assume now the restriction of M to $C_n^!$ is injective:

Then for any integer n the restriction of $\Omega^n M$ to $C_n^!$ is injective.

Let $X \in \mathcal{T}'$. There exists an integer $n \geq 0$ such that $\Omega^n X = Y$, is weakly Koszul and it has a filtration: $Y = U_p \supset U_{p-1} \supset \dots \supset U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, and for all $k \geq 0$, $J^k U_i \cap U_{i-1} = J^k U_{i-1}$. The Koszul modules $G_{B_n^!}(U_i/U_{i-1}) = V_i$ are of Z -torsion and each $Z^j V_i$ is Koszul.

Set $N = \Omega^n M$, the restriction of N to $C_n^!$ is injective.

There exist exact sequences:

$$\begin{aligned} 0 \rightarrow F_{B_n}(Z^{k_i} V_i)[-k_i] \rightarrow F_{B_n}(Z^{k_i-1} V_i/Z^{k_i} V_i)[-k_i+1] \dots \\ \rightarrow F_{B_n}(V_i/ZV_i) \rightarrow U_i/U_{i-1} \rightarrow 0 \text{ where each } F_{B_n}(Z^j V_i/Z^{j+1} V_i) \cong B_n^! \otimes_{C_n^!} X_{ij} \end{aligned}$$

an induced module of a Koszul $C_n^!$ -module X_{ij} .

The exact sequences: $0 \rightarrow B_n^! \otimes_{C_n^!} X_{ik_i} \rightarrow B_n^! \otimes_{C_n^!} X_{ik_i-1} \rightarrow K_{k_i-2} \rightarrow 0$ induce exact sequences:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{B_n^!}(K_{k_i-2}, N) \rightarrow \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} X_{ik_i-1}, N) \rightarrow \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} X_{ik_i}, N) \\ \rightarrow \text{Ext}_{B_n^!}^1(K_{k_i-2}, N) \rightarrow \text{Ext}_{B_n^!}^1(B_n^! \otimes_{C_n^!} X_{ik_i-1}, N) \rightarrow \text{Ext}_{B_n^!}^1(B_n^! \otimes_{C_n^!} X_{ik_i}, N) \\ \rightarrow \text{Ext}_{B_n^!}^2(K_{k_i-2}, N) \rightarrow \text{Ext}_{B_n^!}^2(B_n^! \otimes_{C_n^!} X_{ik_i-1}, N) \rightarrow \dots \end{aligned}$$

where $\text{Ext}_{B_n^!}^j(B_n^! \otimes_{C_n^!} X_{ik_i-l}, N) \cong \text{Ext}_{C_n^!}^j(X_{ik_i-l}, N) = 0$ for all $j \geq 1$.

It follows: $\text{Ext}_{B_n^!}^j(K_{k_i-2}, N) = 0$ for all $j \geq 2$. But $\text{Ext}_{B_n^!}^j(K_{k_i-2}, N) \cong \text{Ext}_{B_n^!}^{j-1}(K_{k_i-2}, \Omega^{-1} N)$ and $\text{Ext}_{B_n^!}^j(K_{k_i-2}, \Omega^{-1} N) = 0$ for all $j \geq 1$.

The sequences: $0 \rightarrow K_{k_i-2} \rightarrow B_n^! \otimes_{C_n^!} X_{ik_i-2} \rightarrow K_{k_i-3} \rightarrow 0$ induce exact sequences:

$$\begin{aligned} \text{Ext}_{B_n^!}^j(K_{k_i-3}, \Omega^{-1} N) \rightarrow \text{Ext}_{B_n^!}^j(B_n^! \otimes_{C_n^!} X_{ik_i-2}, \Omega^{-1} N) \rightarrow \text{Ext}_{B_n^!}^j(K_{k_i-2}, \Omega^{-1} N) \\ \rightarrow \text{Ext}_{B_n^!}^{j+1}(K_{k_i-3}, \Omega^{-1} N) \rightarrow \text{Ext}_{B_n^!}^{j+1}(B_n^! \otimes_{C_n^!} X_{ik_i-2}, \Omega^{-1} N) \rightarrow \dots \end{aligned}$$

Therefore $\text{Ext}_{B_n^!}^{j+1}(K_{k_i-3}, \Omega^{-1} N) = 0$ for $j \geq 1$ which implies

$$\text{Ext}_{B_n^!}^j(K_{k_i-3}, \Omega^{-2} N) = 0 \text{ for } j \geq 1.$$

Continuing by induction there exist some $m \geq 0$ such that

$$\text{Ext}_{B_n^!}^j(U_i/U_{i-1}, \Omega^{-m} N) = 0 \text{ for } j \geq 1.$$

By induction on p we obtain $\text{Ext}_{B_n^!}^j(Y, \Omega^{-m} N) = 0$ for $j \geq 1$, in particular $\text{Ext}_{B_n^!}^1(Y, \Omega^{-m} N) = 0$.

By Auslander-Reiten formula, $\text{Ext}_{B_n^!}^1(Y, \Omega^{-m} N) \cong D(\underline{\text{Hom}}_{B_n^!}(\Omega^{-m} N, \Omega^2 Y)) \cong D(\underline{\text{Hom}}_{B_n^!}(N, \Omega^{2+m} Y)) \cong D(\underline{\text{Hom}}_{B_n^!}(\Omega^n M, \Omega^{2+m} Y))$.

It follows $\underline{\text{Hom}}_{B_n^!}(\Omega^n M, \Omega^{2+m} X) = 0$ which implies $\underline{\text{Hom}}_{B_n^!}(M, \Omega^{2+m} X) = 0$.

Observe m depends only on X . Taking $\Omega^{2+m} M$ instead of M we obtain $\underline{\text{Hom}}_{B_n^!}(\Omega^{2+m} M, \Omega^{2+m} X) = \underline{\text{Hom}}_{B_n^!}(M, X) = 0$.

Therefore $\underline{\text{Hom}}_{gr B_n^!}(M, X) = 0$. It follows $M \in \mathcal{F}'$. \square

Theorem 5. *The category \mathcal{F}' is closed under the Nakayama automorphism, \mathcal{F}' has Auslander Reiten sequences and they are of the form $\mathbb{Z}A_\infty$. Moreover, \mathcal{F}' is a triangulated category with Auslander-Reiten triangles and they are of type $\mathbb{Z}A_\infty$.*

Proof. Let M be an indecomposable non projective object in \mathcal{F}' and $0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$ exact with P the projective cover of M . Since the restriction of P to $C_n^!$ is projective and restriction is an exact functor, it follows ΩM is in \mathcal{F}' . Similarly, $\Omega^{-1}M$ is in \mathcal{F}' .

If P is a projective $B_n^!$ -module and σ the Nakayama automorphism, then σP is also projective. Therefore: \mathcal{F}' is closed under the Nakayama automorphism.

It is clear now that \mathcal{F}' has Auslander Reiten sequences and they are of the form $\mathbb{Z}A_\infty$, by [MZ].

Let $f: M \rightarrow N$ be a homomorphism with M, N in \mathcal{F}' and let $j: M \rightarrow P$ be the injective envelope of M . There is an exact sequence: $0 \rightarrow M \rightarrow P \oplus N \rightarrow L \rightarrow 0$ with M and $P \oplus N$ in \mathcal{F}' . Then L is also in \mathcal{F}' and the triangle $M \rightarrow N \rightarrow L \rightarrow \Omega^{-1}M$ is a triangle in \mathcal{F}' . \square

We have characterized the pair $(\mathcal{F}', \mathcal{T}')$ corresponding to $(\mathcal{T}, \mathcal{F})$ under the duality $\bar{\phi}: \underline{\text{gr}}_{B_n^!} \rightarrow D^b(\text{Qgr}_{B_n})$. Applying the usual duality $D: \underline{\text{gr}}_{B_n^!} \rightarrow \underline{\text{gr}}_{B_n^! \circ p}$ we obtain a pair $(D(\mathcal{T}'), D(\mathcal{F}'))$ which corresponds to $(\mathcal{T}, \mathcal{F})$ under the equivalence:

$$\bar{\phi} D: \underline{\text{gr}}_{B_n^! \circ p} \rightarrow D^b(\text{Qgr}_{B_n}).$$

Observe the following:

From the bimodule isomorphism $B_n^! \sigma^{-1} \cong D(B_n^!)$, for any induced $B_n^!$ -module $B_n^! \otimes_{C_n^!} X$, there are natural isomorphisms:

$$\begin{aligned} \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} X, D(B_n^!)) &\cong D(B_n^! \otimes_{C_n^!} X) \cong \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} X, B_n^! \sigma^{-1}) \cong \\ \text{Hom}_{C_n^!}(X, B_n^! \sigma^{-1}) &\cong \text{Hom}_{C_n^!}(X, C_n^!) \otimes_{C_n^!} B_n^! \sigma^{-1}. \end{aligned}$$

For any finitely generated right $C_n^!$ -module Y there exists a left $C_n^!$ -module X such that $\text{Hom}_{C_n^!}(X, C_n^!) \cong Y$, hence $D(B_n^! \otimes_{C_n^!} X) \sigma \cong Y \otimes_{C_n^!} B_n^!$. Since \mathcal{T}' is invariant under σ , $D(\mathcal{T}')$ is also invariant under σ and $D(\mathcal{T}')$ contains the induced modules.

Let B be a triangulated subcategory of $\underline{\text{gr}}_{B_n^! \circ p}$ containing the induced modules. A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in $\underline{\text{gr}}_{B_n^!}$ comes from an exact sequence

$$0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ u \end{pmatrix}} B \oplus P \xrightarrow{(g,v)} C \rightarrow 0 \text{ with } P \text{ a projective module, hence } D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \rightarrow D(C)[1] \text{ is a triangle in } \underline{\text{gr}}_{B_n^! \circ p}.$$

Therefore: $D(B)$ is a triangulated category containing the duals of the induced modules $D(Y \otimes_{C_n^!} B_n^!) \cong$

$$D(D(B_n^! \otimes_{C_n^!} X) \sigma) \cong D(\text{Hom}_{C_n^!}(X, C_n^!) \otimes_{C_n^!} B_n^!) \cong \text{Hom}_{B_n^!}(\text{Hom}_{C_n^!}(X, C_n^!) \otimes_{C_n^!} B_n^!, \sigma B_n^!) \cong \sigma B_n^! \otimes_{C_n^!} X.$$

Clearly $\sigma D(B)$ is a triangulated category containing the induced modules. Therefore: $\mathcal{T}' \subset \sigma D(B)$. Since \mathcal{T}' is closed under Nakayama's automorphism σ , $\mathcal{T}' \subset D(B)$. It follows $D(\mathcal{T}') \subset B$ and $D(\mathcal{T}')$ can be described as the smallest triangulated subcategory of $\underline{\text{gr}}_{B_n^! \circ p}$ that contains the induced modules.

The usual duality D induces an isomorphism:

$$D: \text{Ext}_{C_n^!}^i(M, N) \rightarrow \text{Ext}_{C_n^! \circ p}^i(D(N), D(M)).$$

It follows that the restriction of M to $C_n^!$ is injective if and only if the restriction of $D(M)$ to $C_n^{op!}$ is projective (injective). It follows $D(\mathcal{F}')$ is the category of $B_n^{op!}$ -modules whose restriction to $C_n^{op!}$ is injective.

$T=D(\mathcal{T}')$ is a "epasse" subcategory of $\underline{\text{gr}}_{B_n^{!op}}$. The functor $\bar{\phi}D$ induces an equivalence of categories: $\underline{\text{gr}}_{B_n^{!op}}/T \cong D^b(\text{Qgr}_{B_n^{op}})/\mathcal{T}$ and we proved $D^b(\text{Qgr}_{B_n^{op}})/\mathcal{T} \cong D^b(\text{gr}_{(B_n)_Z})$.

The equivalence $\text{gr}_{(B_n)_Z} \cong \text{mod}_{A_n}$ induces an equivalence: $D^b(\text{gr}_{(B_n)_Z}) \cong D^b(\text{mod}_{A_n})$.

We have proved:

Theorem 6. *There is an equivalence of triangulated categories:*

$$\underline{\text{gr}}_{B_n^!}/T \cong D^b(\text{mod}_{A_n}).$$

The category $F=D(\mathcal{F}')$ is the category of all T -local objects it is triangulated. By [Mi], there is a full embedding: $F \rightarrow \underline{\text{gr}}_{B_n^{!op}}/T \cong D^b(\text{mod}_{A_n})$.

Proposition 7. *The category $\text{ind}_{C_n^!}$ of all induced $B_n^!$ -modules is contravariantly finite in $\text{gr}_{B_n^!}$.*

Proof. Let M be a $B_n^!$ -module and $\mu: B_n^! \otimes_{C_n^!} M \rightarrow M$ the map given by multiplication. Let $\alpha: \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} M, M) \rightarrow \text{Hom}_{C_n^!}(M, M)$ the morphism giving the adjunction. It is easy to see that $\alpha(\mu)=1_M$.

Let $\varphi: B_n^! \otimes_{C_n^!} N \rightarrow M$ be any map and $\alpha(\varphi)=f: N \rightarrow M$ the map given by adjunction. There is a commutative square

$$\begin{array}{ccc} \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} M, M) & \xrightarrow{\alpha_M} & \text{Hom}_{C_n^!}(M, M) \\ \downarrow (1 \otimes f, M) & & \downarrow (f, M) \\ \text{Hom}_{B_n^!}(B_n^! \otimes_{C_n^!} N, M) & \xrightarrow{\alpha_N} & \text{Hom}_{C_n^!}(N, M) \end{array}$$

from the commutativity of the diagram $f=\alpha_N(\varphi)=\alpha_N(\mu 1 \otimes f)$ implies $\varphi=\mu 1 \otimes f$.

We have proved the triangle:

$$\begin{array}{ccc} & B_n^! \otimes_{C_n^!} N & \\ & \downarrow \phi & \\ B_n^! \otimes_{C_n^!} M & \xrightarrow{\mu} & M \end{array}$$

commutes. □

Corollary 6. *$\text{add}(\text{ind}_{C_n^!})$ is contravariantly finite.*

Corollary 7. *$\text{ind}_{C_n^!}$ is functorially finite.*

Proof. It is clear from the duality $D(\text{ind}_{C_n^!}) \cong \text{ind}_{C_n^{op!}}$ □

Observe $(\text{ind}_{C_n^!})^\perp \cong \mathcal{F}'$ however $\text{ind}_{C_n^!}$ is not necessary closed under extensions and we can not conclude \mathcal{F}' contravariantly finite.

For the notions of contravariantly finite, covariantly finite and functorially finite, we refer to [AuS].

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